LEARNING THE MULTIPLICATION COMBINATIONS

by

WILLIAM A. BROWNELL

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BY

WILLIAM A. BROWNELL

AND

DORIS V. CARPER

DUKE UNIVERSITY PRESS
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W. A. B.

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LEARNING THE MULTIPLICATION COMBINATIONS

CHAPTER I

PURPOSE AND PLAN OF THE MONOGRAPH

This monograph is intended to serve four purposes: (1) to review previously reported research relating to the learning and the teaching of the multiplication combinations; (2) to present two new and rather extensive studies on the learning of these combinations; (3) to consider critically the concept of "readiness" as this concept applies to learning the multiplication combinations and to report new data on the problem; and (4) to suggest a program of instruction, based partly on research and partly on theory, for the teaching of these combinations.[1]

A separate "part" of the monograph is set aside for each of the four purposes. Accordingly, in Part I previously reported research on the teaching of the combinations will be critically examined. Twenty-nine studies will be classified under appropriate headings, and the experimental procedures, findings, and recommendations will be considered with respect to their validity and value. (The research on "readiness," however, is relegated to Chapters VIII and IX of Part III, where the topic "readiness" alone is treated.)

To look ahead a bit, it may be noted now that previously reported research has been limited almost wholly to questions relating to the methodology and organization of instruction. In other words, exceedingly few data have been reported with regard to the way in which children actually learn the combinations. This statement paves the way for Part II, which will contain accounts and results of two original studies. In these studies the principal point of interest was precisely that of the changes in behavior which take place as children advance to mastery of the combinations. The hypothesis is that teaching procedures can only by accident be of the best unless they are selected and directed according to the way in which children learn.

---

[1] In educational literature one encounters four ways of denoting the number of multiplication combinations to be taught. The numbers are: (a) 100, (b) 55, (c) 81, and (d) 45.

(a) If each digit from 0 to 9 is taken as multiplier and each is also taken as multiplicand, one hundred combinations result (from 0 × 0 to 9 × 9). (b) Ten of the one hundred combinations are "doubles" (e.g., 1 × 1, 2 × 2); and the other ninety combinations yield forty-five pairs (e.g., the combination 3 × 2 and its reverse 2 × 3 give one pair). Estimated in this way, there are fifty-five combinations. (c) Sometimes the nineteen 0-combinations (e.g., 0 × 8, 4 × 0) are regarded as a special group, and the remaining eighty-one are taught separately. (d) When the paired combinations among the eighty-one combinations in (c) are taught together, the expression "the forty-five combinations" is appropriate.

When in this monograph it is important to recognize experimental or pedagogical differences in the number of combinations to be learned or taught, the reader will be advised that reference is to the one hundred combinations (a), or to one of the other numbers (b), (c), or (d). In this monograph, too, the terms "combinations" and "facts" will be used interchangeably, in spite of the preference for the former term on the part of some students of arithmetic (e.g., Wheat). The term "facts" has good standing in educational literature, and a certain amount of variety in language seems desirable.

Part III will be devoted to "readiness" for the combinations. The problem of "readiness" is a relatively new one (ten or fifteen years old) in educational thought; but while the idea of "readiness" is new, it bulks large in contemporary curriculum planning. It is sometimes recommended that the multiplication combinations, all or part of them, be postponed from their traditional place in Grade III to Grade IV or even later. The research basis for this new practice will be subjected to criticism, and new data on the problem will be presented in Chapters VIII and IX.

In a way, Parts I, II, and III are but preparatory to Part IV, which is more evidently "practical" in nature. Perhaps the average teacher will be interested in Chapter X alone, for here he will find a suggested program for teaching the combinations. But it is hoped that he will find considerable to interest him as well in the earlier chapters. After all, pedagogical practice can be based upon reliable research findings only to the extent that they exist. When research is lacking, practice must be based upon such evidence as can be deduced from sound learning theory.

Part I

Previously Reported Research on Learning and Teaching the Multiplication Combinations

CHAPTER II

CRITICAL ANALYSIS AND SUMMARY OF PUBLISHED INVESTIGATIONS

Of the twenty-nine educational investigations relating to the multiplication combinations, all except five (**3, 7, 10, 11, 15**[1]) have been published since 1920, and more than half since 1930. Twenty-six of the studies contain original quantitative data; one (**21**), a learning study, has been included in spite of absence of supporting data because of its suggestiveness; and two others (**17, 18**) are critical discussions of earlier investigations. A rather thorough canvass of published educational research disclosed no other study which should be added to the list.

The twenty-nine studies, when classified according to the main problem investigated, fall into three groups: (1) methods of teaching, with six studies; (2) psychological aspects of learning, with seven studies; and (3) comparative difficulty of the multiplication combinations, with eighteen studies (two duplicates). In few of the reports is the scope actually limited to a single problem. Furthermore, all three main groups may easily be analyzed into subgroups. For example, the studies in group (2) deal with such diverse problems as the extent of transfer in learning, readiness, brightness and dullness as factors in learning, and so on. Nevertheless, the grouping adopted provides a convenient organization for the purposes of this review.

METHODS OF TEACHING

As has already been suggested, several studies in the second and third groups are not without significance for the methodology of instruction. Thus, for example, Norem and Knight (**14**) describe the methods by which they taught their subjects the combinations, and their findings are to some extent contingent upon the instructional method they used. Their greater concern with the comparative difficulty of the combinations, however, justifies the placement of their report in the third group. Four of the six studies considered in this section clearly belong in group (1) alone. The other two (**8, 27**) are included in group (1) and in group (3) as well, for the experimental interest was about equally divided between methods of teaching and the relative difficulty of the facts. The six studies to be considered here in chronological order are those reported by Mead and Sears (**11**), Fowlkes (**8**), Thiele (**21**), Morrison (**12**), Wheat (**26**), and Wheeler (**27**).

---

[1] Numbers in parentheses, as here, refer to the separate items in the chapter bibliography. See pp. 40-42.

Mead and Sears (**11**) investigated the possibility (a) of teaching the subtraction combinations as extensions of the addition combinations and (b) of teaching the division combinations as extensions of the multiplication combinations. It is the latter part of their study which is pertinent to the present purpose.

The combination $3/\overline{15}$ may be dealt with through multiplication ("three times what are fifteen?"), in what Mead and Sears call "multiplicative division," or it may be dealt with through division ("three into fifteen goes how many times?"), which Mead and Sears designate the "into method." The problem posed for investigation was: Which form of presentation results in the more economical learning?

Two third-grade classes, each numbering perhaps forty pupils (the number is not given), first learned the multiplication facts. Then they were divided into two experimental groups, the one studying the division facts by the method of multiplicative division and the other by the "into method." Thirty minutes a day were spent on the new combinations, and learning was assessed by means of written tests, all of which (including the pretest) were limited to a period of one minute. The pretest, which might well have been made up of the multiplication combinations, consisted actually of addition combinations. It was administered in February. All subsequent tests, March 9, April 6, May 4, June 1, and June 17, were restricted to the division facts, except that on the last date an extra test on division with four-figure dividends was also administered. After each test the pupils were told their errors, in order to permit concentrated study thereon.

The results obtained are inconclusive: the median of the "into" group was 4.3 combinations lower than that of the other group on the June 17 test on the facts alone, but was .7 combination higher on the longer examples. Even had larger and more consistent differences been found, it is doubtful whether experimental conclusions could have meant anything. The groups were probably too small to assure reliable results; the shortness of each testing period must inevitably have introduced many chance variations irrelevant to the experimental problem; all disparities in scores on the division tests were "corrected" for initial differences on the pretest involving addition facts; and nothing is known about the program of instruction apart from the different ways in which the two groups were taught to think of the division facts.

In all probability most of the experimental limitations of the Mead and Sears study may be ascribed to its early date; the report appeared in 1916. At that time quantitative research was just getting under way in education. But whatever the limitations, this investigation may have had important influence. Certainly the practice of teaching the division facts through the multiplication facts has never enjoyed wide popularity as compared with the practice of teaching the subtraction facts through the addition facts.

Fowlkes (8) states that the purpose of his research was "(1) to teach the one hundred combinations by means of test material only, the teacher doing as little as possible; (2) to make remedial adjustments by means of printed directions and devices other than oral instructions; and (3) to ascertain the relative difficulty of the one hundred combinations when they were developed by the pupils individually, and when the frequency of drill on each combination was the same." Because of his twofold interest Fowlkes's investigation is considered here (methods of teaching) as well as later, in connection with the problem of the relative difficulty of the combinations.

Thirty-one Grade IIIB pupils with a median IQ of 104.5 (range, 94 to 137) were taught the one hundred combinations in a period of twenty days. A pretest showed that the median child could give correct answers for twenty-three combinations (range, 0 to 66). When instruction began, ten new combinations were presented each day. (The order of the combinations is not stated, nor does the report tell whether tables were employed.) The daily schedule was as follows: development of the new facts additively, 20 minutes; practice on the new facts on sheets which presented each fact ten times, 10 to 15 minutes; practice on the new facts in problems, 5 to 10 minutes. On the eleventh day a test was given on the one hundred facts; and for the next nine days 10 minutes a day were spent by each child on the combinations he had missed on that test or on later tests.

The following abbreviated table is a sample of Fowlkes's findings:

Number of combinations answered correctly, by days:

	11	*12*	*15*	*17*	*19*	*20*
Median	59.5	66	69	75	88	91
Q_3	70	77.5	81	90	96	98
Q_1	42	54	57	67	67	79

The success achieved by Fowlkes's subjects was considerable. While they attained nothing like "100% mastery," the average pupil could at the conclusion of the experiment write correct answers for ninety-one facts, and three fourths of the children for seventy-nine or more facts. This high degree of achievement is all the more noteworthy because of the shortness of the period of learning. Fowlkes suggests—and most teachers and students of arithmetic would agree with him—that better results would probably have been attained, had the experiment lasted twice as long, and had twenty days instead of ten been taken to develop the facts at the rate of five a day. The success that was achieved was probably the consequence of several factors, among them (a) the attempt to develop meanings before drill (a practice much commoner now than at the time of this study), (b) the systematic nature of the drill materials, and (c) the provision for individual diagnosis and self-directed remedial instruction.

Thiele (**21**) was interested in developing a method of teaching based upon the idea "of providing experiences which will cause pupils to sense number relationships and thereby enable them to develop a method of attack rather than leave this to chance," the purpose being to "shift the emphasis from excessive drilling to study based upon understanding." His article contains quantitative data only with respect to the addition and the subtraction facts (the data on the addition combinations have since been amply confirmed by his more recent research[2]), but his observations with respect to the teaching of the multiplication facts are of value.

Thiele had the subjects in his informal experiment first build the multiplication tables by addition and then study the completed tables for the relationships which they could find. These relationships, expressed as generalizations, were then employed actively by the children in learning the facts themselves. Illustrative facts for the table with 9 as multiplier are: the first figures of the products increase by 1 (18, 27, 36, etc.); the second figures decrease by 1; the sum of the product figures is always 9; 9 × 9 is 81, because 9 × 10 is 90. Many children used the combination 9 × 5 as a center for organizing less well-known facts, e.g., 9 × 6, 4 × 9, etc. Once such relationships had been discovered and used temporarily in thinking products, the children were encouraged by games, flashcard drills, and similar devices to master the facts directly. According to the teachers in the experiment, much less of this repetitive practice was needed for mastery than is the case when relationships and generalizations are not made use of.

---

[2] C. L. Thiele, *The Contribution of Generalization to the Learning of the Addition Facts* (Teachers College, Columbia University Contributions to Education, No. 763. New York: Bureau of Publications, Teachers College, Columbia University, 1938). 84 pp.

It is unfortunate that Thiele has not published his quantitative data. His teaching practices differ greatly from what is customarily done in arithmetic classes, and some teachers will not be convinced of their merits until they have more evidence than they now have. In the absence of the needed data, however, it is not being overly credulous to accept Thiele's "generalizations" as sound aids in learning. The justification for this acceptance lies in the data reported in his comparable experiments with the addition and subtraction facts. In these experiments the subjects who learned by procedures corresponding to those described above for the multiplication facts showed great superiority over subjects who had been taught in the traditional way.

Thus far, three studies relating to methods of teaching have been considered. The fourth is by Morrison (12), whose control-group experiment was conducted in Grade IV. It was apparently in this grade (the account is unclear at this point) that her subjects were first introduced to the multiplication facts. Sixty-two pupils were taught by "mass" methods with emphasis on large-group activities—flashcard drills, games, some written work, and an unexplained "device for crediting children with mastery of the tables." Two other classes numbering seventy-five children were taught by the "individual" method. The chief characteristics of the latter method were: use of individual learning or drill books and of individual cards to show each child his errors, checking of work first by the pupil and then by the teacher, and a high degree of motivation based upon interest in self-improvement. The Wilson Process Test in Multiplication SP was used as a pretest in October, as a final test in March, and as a retest in Grade V in the following September. This test includes eighty-seven of the one hundred facts, forty-one occurring once, thirty-five twice, and eleven three times, as well as more difficult examples in multiplication computation.

Morrison's findings are summarized briefly below:

	Medians		Wilson Process Test		
	CA	IQ	Pretest (Oct.)	End Test (March)	Retest (Sept.)
Mass Practice Group	9 9.3	105.5	18.0	84.8	64.0
Individual Practice Group ...	9 5.9	103.4	8.3	80.1	68.0

Morrison points out that the individual practice group, in spite of initial disadvantages in CA, IQ, and multiplication ability (pretest), made greater immediate gains (71.9 per cent as compared with 66.8 per cent, October to March) and greater permanent gains as shown in the September scores (59.6 per cent as compared with 46.0 per cent) besides having smaller summer losses. She further states, without explanation, that her analysis of errors showed "the usual results."

The program of instruction in the Morrison research had two rather unusual features, first, late introduction to the basic facts, and, second, a very rapid pace of instruction, such that within five or six months the children not only learned the facts but also learned to compute with multipliers as large as four-place numbers. The result was a rather low standard of work in both groups. The advantages claimed for the individual group in the end test and retest seem not to be large. Whether or not the differences are reliable cannot be determined from the published data.

Just what was done in this study to teach the multiplication combinations cannot be ascertained from the article. If one were to infer the procedure from the use of printed practice books in the one group and of much rapid-fire mass practice in the other, one would surmise that in both groups learning was left largely to repetition, with but a modicum of meaning provided ahead of time. If these inferences be valid, the instructional procedures differed considerably from those used by Thiele. But whatever the methods used, it is impossible to tell the differential effects of the two experimental types of instruction upon the learning of the multiplication facts alone.[3]

The fifth study on methods of teaching has been briefly reported by Wheat (**26**), an investigation made by one of Professor Wheat's graduate students, P. G. Michael. Both in general purpose and in experimental technique this research resembles that of Thiele. Neither Thiele nor Michael used a control group, but endeavored simply to discover what happened when they taught the facts according to a novel program of instruction; and both investigators made serious attempts to envelop the combinations with meaning before

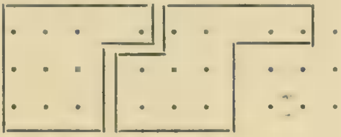
[3] A minor aspect of the Washburne and Vogel study (**25**), summarized later, is a comparison of "individual" and "mass" methods of instruction. The 244 third-grade children taught by the Winnetka "individual" method averaged 2.2 per cent errors on the combinations as compared with 5.1 per cent made by Clapp's pupils (**6**). The presumption is that Clapp's subjects were drawn from schools in which group rather than individual methods of instruction prevailed.

requiring children to master the abstract formulas by repetitive practice. Thiele strove for understanding by stressing mathematical generalizations; Wheat, or Michael, by employing objective representations of the combinations. In the latter study, however, only the thirty-nine supposedly most difficult combinations were taught.

Cards were prepared with the abstract combinations on the front and the corresponding number pictures on the reverse. Two samples are:

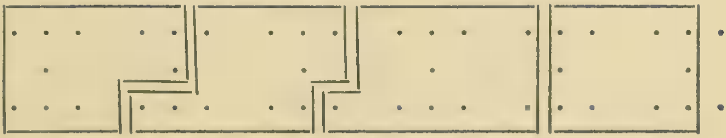
(Front) (Reverse)

$$\begin{array}{r} 9 \\ \times 3 \\ \hline \end{array}$$



(Front) (Reverse)

$$\begin{array}{r} 7 \\ \times 6 \\ \hline \end{array}$$



As shown above, on the reverse of the cards the multiplicands (9 in 3 × 9; 7 in 6 × 7) were pictured in regular groups which were repeated as often as required by the multipliers (three 9's, six 7's). The objects of these groups were then rearranged to make as many new groups of 10 each as possible. Thus, 3 × 9 gave two groups of 10, with 7 over (27); six 7's gave four 10's, with 2 over (42). Pupils studied these cards under conditions which approximated self-instruction, working with the number pictures until they understood (a) the process of multiplication, (b) how this process applied in the case of a given fact, and (c) how the original groups, rearranged into 10's (the base of the Hindu-Arabic system of number notation), yielded their products. They then worked with the abstract combinations on the fronts of the cards, referring to the number pictures only when they could not recall the products. Wheat states that the forty third-grade subjects learned all the difficult combinations with an average of twenty-one trials per combination, and the implication is that this number of trials was much fewer than is usually necessary. Equally important with their success in learning the combinations, according to Wheat, was the fact that the subjects "developed a degree of personal pride in their ability to depend upon themselves."

This feeling of pride in independent methods of attack is unquestionably a desirable feature in learning the multiplication facts, or in learning anything at all, for that matter; but it is to be questioned whether this outcome can be achieved only through use of the particular device used in the Wheat study. There is no a priori reason, for example, why the same self-dependence might not be engendered by the use of Thiele's generalizations. This hypothesis can be confirmed or refuted of course only through other studies. From Wheat's study it is possible only to conclude that pride in independent work habits can be achieved by the pictorial device. It cannot be told how this device compares in effectiveness with other devices at this point; nor can it be told how useful the device as such is for teaching the facts as compared with other instructional procedures.[4]

The most recent study on methods of teaching is that made by Wheeler (27). (As has already been explained, Wheeler was about as much interested in the relative difficulty of the combinations; hence his data on the latter problem are deferred to the third section of this chapter.)

Wheeler's research involved 342 children just beginning Grade III, whose median IQ was 101, and median CA, nine years no months. His program of instruction was frankly and avowedly one of drill. At the start of instruction "a brief explanation was given as to the nature of multiplication and how it differs from addition," and a few comments were made on "the generalizations governing multiplying by zero and 1." From this time on, learning was left to the activities incidental to playing the author's game, MULT-O, which is based upon the well-known game LOTT-O. The method of learning was that of memorization. Counting-up (e.g., thinking "5, 10, 15" to secure the product for 3 × 5), repeated addition ("5 and 5 are 10, and 5 are 15"), and the use of tables were all either withheld or discouraged when observed. The aim was to secure automatic mastery with the greatest economy possible. Twenty minutes a day for four school weeks were thus spent in playing MULT-O. Learning was checked at the end of each week by means of a written test on ninety-nine combinations (0 × 0 is not used in MULT-O); and the same test was used as a pretest.

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[4] Certain criticisms can be made of Wheat's pictured groups, but these criticisms are perhaps best reserved to the last chapter of the monograph, where it will be possible to see how these objective representations can be fitted into a more comprehensive program of instruction.

The results were as follows:

	Tests				
	1	2	3	4	5
Number of cases	342	330	318	307	304
Median score	9.3	30.7	38.1	44.4	51.6
Q_1	1.6	16.7	18.6	27.1	34.6
Q_3	28.2	49.1	54.6	61.6	67.7
Gain per week	...	21.4	7.4	6.3	7.2
Gain per day	...	4.3	1.5	1.3	1.4

Wheeler regards the progress made by his subjects as entirely satisfactory and as evidencing the advantages of his drill program. Yet, the most obvious fact about his results is that he did not attain his end: at the conclusion of the four weeks his pupils were still a long way from mastery of the combinations. The average child "knew" only about half the facts, and a quarter of them "knew" only slightly more than a third of the facts. The poverty of these results is especially clear when they are compared with those obtained in two studies which have already been reviewed in this section. Fowlkes's subjects, like Wheeler's, spent twenty learning periods on the combinations, but at the end of that time the average child could give correct answers for ninety-one instead of fifty-two facts. Wheat's subjects spent an unstated length of time ("twenty-one practices") on thirty-nine difficult facts, but certainly not so long a period as was spent by Wheeler's subjects, and at the end of their learning they could give correct answers for all the studied facts.

Wheeler recognizes that the learning of his subjects was incomplete and estimates that lengthening the period of study by five or six weeks would have produced a reasonable standard of achievement. This estimate is based upon the assumption that the weekly gain of about seven combinations would have continued. There is, however, no reason to accept this assumption. On the contrary, *if* his subjects used exclusively the learning procedures supposedly required by MULT-O, a decline in rate of progress would almost certainly have set in, for they would have experienced steadily mounting trouble in keeping straight the accumulating facts they were expected to know.

Whether or not this objection is sound cannot be determined from Wheeler's data, but the hypothesis is as warranted as is Wheeler's, that a continuance of the period of drill would have yielded the results he sought. It is, in other words, rather dangerous to extrapolate his data for four weeks in order to predict what would have happened in ensuing weeks.

Yet, it is not impossible that Wheeler's subjects could have learned the combinations in a total of nine or ten weeks. If they had done so, they would have achieved success *in spite of,* and not because of, the instructional methods to which they were exposed. Success would have come to the extent that they were able to discover and use the many relationships which Wheeler so carefully kept from them. Wheeler himself testifies to the value of the "simple generalizations" taught at the start of the experiment; and the large gain in the first week (twenty-one combinations) is tangible evidence of their merits. Wheeler also states that his subjects made profitable use of the principle governing reverses, though they apparently discovered this principle for themselves.

The truth of the matter is that in his experiment Wheeler learned very little about his subjects' methods of mastering the combinations. He knew how he had the children taught, but he did not find out what they did in response to these teaching methods. The foregoing sentences imply that children may learn by processes quite unlike those which are supposed to be set up by instruction. They do, as the data in Part II of this monograph will amply demonstrate. Wheeler might well have supplemented the data from his group tests with observations concerning children's thought processes before recommending the wholesale and virtually exclusive adoption of drill methods to teach the multiplication facts.

Summary with Regard to Methods of Teaching

The six studies on methods of teaching the multiplication combinations warrant the following conclusions:

1. There is no convincing evidence that pupils should employ their multiplication combinations, by extension, to cover the division facts as well. On the other hand, there is no convincing evidence that they should not do so. The one research study on the problem does not provide unambiguous data. Nevertheless, the practice has become general to teach the division facts as such either with or after the multiplication facts.

2. There is plenty of evidence (Fowlkes, Thiele, Wheat) that children can learn the multiplication combinations in Grade III, provided that appropriate and wise methods of instruction are used. (The word "learn" in this connection means only "can give correct answers for"; but this meaning of the term has traditionally been the criterion of satisfactory achievement.)

3. The research evidence in favor of "individual" as contrasted with "group" methods of teaching is less emphatic in the case of the multiplication facts than it is in other areas of arithmetic and in other subject-matter areas. In general, "individual" methods have the support of psychological theory, for they enable the learner to detect his errors early and to adopt corrective measures at once.

4. There is some evidence that children learn the facts more readily when those facts have first been invested with meaning (Thiele, Wheat). This evidence is as yet incomplete, because the research techniques employed have supplied no comparisons with alternative and nonmeaningful methods.

5. The evidence with regard to the value of drill procedures is not clear, almost wholly because data are lacking to show precisely what drill does to advance learning. Drill methods do seem to produce a degree of proficiency (27), though little is known about the permanence and usefulness of what is learned by these methods. There is evidence, admittedly not in connection with the multiplication facts, that children learn otherwise than by repetition when they do not understand what they are supposed to learn.[5] Part of the apparent success of drill methods may be attributable to causes unrelated, even antagonistic, to such methods.

6. Independence of work habits is an outcome worth seeking in learning the facts. This outcome, it is claimed on the basis of experimental observations (and should, it would seem, be granted by common sense), is best attained when instructional methods lead children to find and employ relationships—in a word, to understand what they learn.

PSYCHOLOGICAL ASPECTS OF LEARNING AND USE

Transfer of Training

The one study in this category (4) was an attempt to determine the extent to which transfer operates, or may operate, from certain learned combinations to other related and unstudied combinations. The investigation was made by R. H. Dahlgren, but is reported by Brueckner, who directed the study, in the form of a short abstract.[6]

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[5] William A. Brownell and Charlotte B. Chazal, "The Effects of Premature Drill in Third-Grade Arithmetic," *Journal of Educational Research,* XXXIX (Sept., 1935), 17-28. See also the four-page supplement containing charts omitted in this article.

[6] In this report but a few of the details of the experimental procedure are described. It is possible, however, to complete the description by referring to a comparable study reported elsewhere by Brueckner. See E. A. Beito and

In this research transfer was measured from the learning of the "direct" combinations (combinations in which the multiplier is the smaller of the two factors)—and these were specifically taught—to the "reverse" combinations (multiplicand the smaller factor), which were not taught. Only the tables with 2, 3, and 4 were used, thus making a total of forty-two different combinations.[7] The subjects were 125 pupils in Grade IIIB and, surprisingly enough, in Grade IIB.

The tables for 2 and for 2's were taught first. In the pretest covering the sixteen combinations, direct and reverse, 802 wrong answers or omissions occurred with both direct and reverse forms, 867 (802 + 65) with direct forms, and 847 (802 + 45) with reverse forms. Following the pretest and on the same day came a period of study on the eight direct forms only, ending with a test limited to the same facts. The number of errors and omissions dropped from 867 to 142. During the next three days the daily test at the beginning of the period contained only direct forms, and these were studied during the period and tested again at the end of the period. On the fifth day, after the test on the direct forms only and after a short period of study on these same forms, a test was given on all sixteen combinations involving 2, thus including the reverse (unstudied) facts as well as the direct forms. The total errors on the direct forms decreased during the week from 867 to 23, an improvement of 97.3 per cent, which was attributed to direct instruction. The corresponding reduction in errors for the reverse forms was 89.7 per cent, and this improvement was explained as the result of transfer.

During the second week of the experiment the tables for 3 and for 3's were dealt with according to the same schedule, and during the third week the tables for 4 and for 4's. The results for all three tables follow in condensed form:

Leo J. Brueckner, "A Measurement of Transfer in Learning of Number Combinations," *Twenty-ninth Yearbook of the National Society for the Study of Education* (Bloomington, Illinois: Public School Publishing Co., 1930), pp. 569-587. This earlier study dealt with transfer in the case of the addition combinations.

[7] There were eight direct and eight reverse facts for 2 (2 × 2 being omitted), seven direct and seven reverse facts for 3 (3 × 3 being omitted), and six direct and six reverse facts for 4 (4 × 4 being omitted). However, in the prosecution of the study some facts already learned were included in the second and the third weeks. In the second week, when the 3-facts were presented, the two learned facts 2 × 3 and 3 × 2 were among those in the tests. In the third week, when the 4-facts were presented, the four learned facts 2 × 4, 4 × 2, 3 × 4, and 4 × 3 were in the tests. The effect thus of including in the later weeks facts which had already been learned is shown in the declining number of errors in the second and third pretests. See the abbreviated table above.

Table	Errors First Day		Errors Last Day	Per Cent Gain
2	Direct,	867	23	97.3
	Reverse,	847	87	89.7
3	Direct,	559	33	94.0
	Reverse,	542	76	85.9
4	Direct,	514	61	88.1
	Reverse,	549	80	85.4

For the three tables combined, the per cents of improvement were 94.0 for the direct facts and 87.5 for the reverse facts. The improvement was therefore regularly greater from direct instruction than from transfer, but by a margin much smaller than would have been predicted by those who tend to minimize the usefulness of transfer in learning. The decline in the amounts of learning both for taught and for untaught combinations from the facts for 2 to the facts for 3 and then for 4 probably reflects the influence of too rapid a pace of instruction—forty-two new combinations in three weeks.

This investigation shows conclusively that transfer may greatly reduce the load of learning the combinations and may do so without explicit effort to this end on the part of the teacher. The only instructional factor in this experiment which could have facilitated transfer was the organization of the facts in the form of tables. With deliberate encouragement to transfer learning and with the provision of generalizations besides those implicit in the tables, even more transfer might be expected.[8]

Nothing in the Brueckner report reveals just what was transferred from the learned to the unlearned facts, nor how the transfer took place. For example, (a) did the pupils transfer their "knowledge" of the learned combinations, or (b) did they discover somehow that direct and reverse combinations have the same answers and then transfer this principle, or (c) did they form the habit of reversing the places of the multiplier and the multiplicand in the untaught combinations and use this practice when they were tested on the reverse combinations?[9]

[8] Experimentation on other problems in the learning of arithmetic has revealed large transfer effects. To mention but a few sample studies: F. B. Knight and A. O. H. Setzafandt, "Transfer within a Narrow Function," *Elementary School Journal,* XXIV (June, 1924), 780-787 (common fractions); Herbert T. Olander, "Transfer of Learning in Simple Addition and Subtraction," *Elementary School Journal,* XXXI (Jan. and Feb., 1931), 358-369, 427-437; James R. Overman, *An Experimental Study of Certain Factors Affecting Transfer of Training in Arithmetic* (Educational Psychology Monographs, No. 29; Baltimore: Warwick and York, 1931). 235 pp.

[9] Data to be presented in Chapters IV and V show that children do have the habit of translating reverse forms to direct forms. Though the extent of

Thought Content in Adult Use of Combinations

One of the earliest among the studies reviewed in this monograph (3) was made by Browne at a time when psychology was greatly concerned with the "content of consciousness." Browne's purpose was to discover how adults deal with various arithmetical tasks, among them the supplying of products for certain familiar multiplication facts. His four subjects were given a pack of twenty cards, each containing some digit from 2 to 9. The experimenter, working with the subjects individually, announced a multiplier (2 to 9) just before the signals "Ready" and "Go," after which the subject drew a card from the pack and stated the product of the multiplier and the figure on the card. Browne observed all significant aspects of behavior and recorded each subject's description of his procedures.

The nature of the findings is indicated by paraphrasing a sample: Three of the four subjects were much given to "motorizing"; they tended to recite the tables, particularly when the products were large, and even when they were fairly certain of the correctness of their answers; the effort to maintain rhythm in announcing products became at times a distracting influence; the multiplier rather than the multiplicand seemed to determine the relative difficulty encountered from combination to combination.

The use of adult subjects, and especially of so small a number, reduces the significance of the findings for the improvement of instruction. Nevertheless, Browne drew several "pedagogical inferences." Insofar as these "inferences" apply to the teaching of the combinations they are: (a) the tables should be taught as such and thoroughly mastered; (b) children should understand the process of multiplication as a special case of addition; (c) the multiplier should always be stated as the first factor; and (d) such combinations as 9 × 3 should be immediately changed in thought by making the smaller digit the multiplier (3 × 9), thus reducing the number of separate facts to be learned.

It will be noted that not all of these "inferences" follow from the investigation. Still, some of them are not without interest—for example, the statement favoring the use of tables in instruction (a position objected to by many later students of arithmetic) and the use of

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this practice was not great in the investigations in question, still under the conditions of the Brueckner study it could easily have been encouraged. The third possibility above, then, may explain what happened in the case of the Brueckner subjects.

the research techniques of observation and verbal report which for a long time fell into disfavor and have been but lately restored to good standing.

Brightness and Dullness as Factors in Learning

In his investigation (28) Wilson used as subjects thirty "bright" children (IQ between 110 and 120), fifteen of them aged 9 and fifteen aged 12, and thirty "dull" children (IQ 80 to 90), likewise divided fifteen and fifteen between ages 9 and 12. These two groups of children, differing considerably in "intelligence," were required to learn several new tasks, the purpose being to discover similarities and unlikenesses in the learning of bright and dull children.

Among the learning tasks was that of mastering a series of new multiplication combinations, from 2 × 67 to 9 × 67. These combinations were presented on cards in both direct and reverse forms and in both vertical and horizontal orders, the latter with the products given. The results in the end tests for the multiplication facts are sampled in the following table:

Number of Correct Answers

Combination	Dull		Bright		Total
	9	12	9	12	
2 × 67..............	21	93	99	118	331
67 × 2..............	22	98	92	118	330
3 × 67..............	31	74	73	102	280
67 × 3..............	20	74	71	101	266
.	.	.	.	.	.
.	.	.	.	.	.
9 × 67..............	29	95	58	89	271
67 × 9..............	31	99	61	86	277
Total	312	975	745	1,134	3,166

So far as learning is concerned, Wilson's conclusions with regard to the multiplication combinations agree with his conclusions from his other experiments. What was difficult for the bright children was difficult for the dull, and vice versa; "human beings, despite great variations in ability and interests, learn in much the same way one to another"; that is, the general method of attack is similar for bright and dull children, though there are differences in rate, accuracy, and retentiveness.

Wilson's finding that the reverse forms of number facts are no more difficult for learning than the direct forms is debatable in the light of other research evidence. And Wilson's main point, that bright and dull children learn in about the same way, needs to be taken with caution. Wilson specifically allows for unlike rates of learning, among other differences noted, and this unlikeness in rate alone requires large differences in the pace of instruction for bright and dull learners. For example, it is to be expected that dull children will require many more experiences, both of any given kind and of different kinds, to arrive at the same generalizations and understandings that come rather quickly to bright children. The result is that what appears to be merely a quantitative difference in instruction becomes almost a qualitative difference as well.

Knowledge of Familiar and Unfamiliar Combinations Possessed by Adults

Triplett (22) gave to 419 high-school graduates enrolled in a business college a twenty-minute test consisting of 611 multiplication facts. The test included unusual combinations as high as 25 × 25, with a few omissions in the various tables. Instructions were to write the answers without actually solving for unknown products. Triplett reports that the latter instruction was not adhered to, and there is reason to believe that the time limit was too short. The average score for the 293 girls was 214.6 (S.D. 58.1) and for the 126 boys, 217.9 (S.D. 63.2). Other differences, none of them large, were found between students from city and from rural schools. Errors for the combinations above 12 × 12 were not much more numerous than for the combinations below this point.

However valuable for other purposes, the results of this study tell little about the guidance of instruction on the basic multiplication facts in the lower grades. And, indeed, Triplett himself claims no such significance.

Readiness

The published research on "readiness" for the multiplication facts is meager. As a matter of fact, up until recently only one study (24) had been organized specifically to determine when these facts should be taught. There is some indirect evidence in other research, for example, in references 9 and 29, and this monograph contains new data on the problem. All these studies will be considered in Part III. The summary of previous research on readiness is omitted here and is reserved to the later point in order to permit a more unified treatment of the issues involved.

Section Summary

In this section seven investigations have been assembled under the topic "Psychological aspects of learning and use," and have been regrouped under five sub-topics. One sub-topic, readiness, has been postponed to Part III of this monograph. The two studies with adults on the extent of knowledge of the multiplication tables to 25 × 25 (22) and on their thought procedures in dealing with familiar combinations (3) do not seem to be very suggestive for the teaching of the combinations to children in Grade III. So far as the other two sub-topics are concerned, the following conclusions seem to be warranted:

1. There is reason to believe that children can profit by transfer in learning the multiplication combinations. Transfer has been shown to take place (4) in the case of the direct-reverse pairs of facts even when instruction does not call attention to the possibility. In other words, in this case children are apt to hit upon the principle involved themselves and to use it to advantage. It may be inferred (a) that transfer should operate in connection with other relationships among the combinations and (b) that it may be increased considerably in amount if effort is directed to that end.

2. It is true that in a general way dull and bright children learn alike, but the educational implication of this general likeness should not be exaggerated. The statement does not mean that we should teach bright and slow children in the same way. The differences in their rate of work, their accuracy, and their retentiveness (28), to mention but a few variables, mean that instructional methods must be adapted accordingly. Slow children require more experiences of a given kind and a greater variety of experiences than do bright children to arrive at the same level of understanding and degree of mastery.

COMPARATIVE DIFFICULTY OF THE COMBINATIONS

Of all the problems recognized in this summary, the most popular for research has been the relative difficulty of the combinations. Eighteen of the twenty-nine references in the bibliography have been specifically planned to secure difficulty ratings. Sixteen of the references present original data; these will be considered first. The other two references (17, 18) are based upon the results of other studies and will be treated more or less incidentally toward the end of this section.

CHART I

SUMMARY OF STUDIES ON RELATIVE DIFFICULTY OF COMBINATIONS

Investigator	*Date*	*Type*	*Subjects*	*Combinations Studied*	*Criterion(ia) of Difficulty*	*Nature of Report*
Phillips (15)	1897	Status.	723—IX, adults; 440 questionnaires returned.	Figures most difficult in addition and multiplication.	Teachers' opinions; number of errors.	7 reported most difficult by 327; 9 next, 205 mentions.
Döring (7)	1912	Status.	1053—ages 9-14; adults.	Three hardest combinations reported.	Opinions; number of errors.	Ranks of the sixteen combinations reported as hardest; relative difficulty of tables through 10's.
Holloway (10)	1914	(a) Controlled learning.	172—IIA, IIB, IIIA.	(a) 78 "fundamental facts."*	(a) (1) coefficient of difficulty, i.e., total no. errors made in learning divided by total no. errors overcome in learning, the quotient corrected for 50 pupils. (2) total errors coefficient, i.e., all errors made on tests and in practice periods, these totals being divided by number of pupils.	(a) Direct and reverse facts considered together, and facts thus paired ranked from 1 × 1 to 12 × 12.
		(b) Status.	1215—II, III, IV, and V; 384 in Sept., 831 in June.	(b) Same as (a).	(b) (1) total errors in answer. (2) total errors in process (as opposed to answers).	(b) Same as (a).
		(c) Controlled learning.	530—III.	(c) Same as (a).	(c) Comparison of errors in Sept. and in June; effect of 10 weeks' vacation.	(c) Same as (a).
		(d) Status.	315—IIB, IIIA; IVB, IVA.	(d) Same as (a) but grouped into 16 sets.	(d) Time to "think" answers; time to "think" and write answers less time simply to write answers.	(d) Results not treated for separate facts, but for 16 groups of facts; hence no data on relative difficulty of the various facts.
Smith (19)	1921	Status.	42, made up of unstated numbers from III, IV, V, VII and one class in a teachers college.	100 facts.	Time to give products, individually determined by means of tachistoscopic presentation and timing by stop watch; final ranks determined from median of five class medians; total errors also given, but not used as criterion.	Separate ranks for the facts, based exclusively on relative time to give products.
Clapp (5, 6)	1924	Status.	(a) 6223, IV-VIII.	100 facts.	Number of errors on written group test.	Separate ranks for facts, by grades and for total of subjects.
			(b) 2837, IV-VIII.	100 facts appearing in 10 computational examples.	Number of errors on written individual test.	Separate ranks for facts, by grades and for total of subjects.
Batson and Combellich (2)	1925	Status.	83; 1 in VII; 9 in high school; 50 in college; 23 adults with degrees, including 1 with Ph.D.	100 facts.	Principal criterion is time required to find and write products; timing by electric apparatus in units of .2 second; final ranks based upon average time per combination for all subjects. Data on number of errors reported but not regarded as significant.	Time given for each fact ("mental" time, i.e., for thinking; "physiological" time, for writing; and time for both combined); all facts ranked accordingly.

CHART I *(Continued)*

Investigator	*Date*	*Type*	*Subjects*	*Combinations Studied*	*Criterion(ia) of Difficulty*	*Nature of Report*
ReBarker (16)	1926	Status.	4000—V-VII.	100 facts.	Per cents of errors expressed in terms of sigma units; factors given orally, answers written.	Ranks of separate facts by grades and grades combined.
Fowlkes (8)	1927	Controlled learning; 35 min. a day; 20 days; special mimeographed practice.	31—IIIB; IQ's 94-137; median IQ, 104.5.	100 facts.	Number of errors in learning.	Ranking for separate facts.
Anspaugh and Phipps (1)	1928	Status.	1011—VI.	100 facts.	Per cents of error.	Same as above, both numbers and per cents of error given.
Washburne and Vogel (25)	1928	(a) Status.	(a) 758—III-VIII.	(a) 100 facts.	(a) and (b): numbers of errors.	(a) Ranks regardless of grade, method of teaching, or form of test (three forms used).
		(b) Controlled learning.	(b) 399—III.	(b) Same.		(b) Ranks for Grade III only. In both cases only the ranks for the 25 easiest and the 25 hardest are reported.
Norem and Knight (14)	1930	Controlled learning; individual instruction and testing.	25—III; IQ's 98-148; median IQ, 121.	100 facts.	Composite scores, made up of measures of (a) number of practices, (b) number of errors, and (c) time of response as obtained by Jacquet polygraph; composite calculated by combining three measures and weighting them "entirely on a common-sense basis."	Ranking for separate facts on composite basis, plus data for each fact on three bases mentioned, plus "per cent knowing at beginning" and "average number of weekly tests."
Smith and Eaton (20)	1939	Status.	77—IVA and IVB.	100 facts.	Number of errors, the combinations being presented at a set rate by a special apparatus ("number master") and the answers written.	Rankings for separate facts according to number of errors (0-34).
Murray (13)	1939	Uncontrolled learning.	1,775 Scottish children, "completing their second session of the Primary Division"; average age, 9 yrs.	100 facts.	Number of errors.	Rankings for separate facts.
Washburn (23)	1939	Status.	3,800(?)—IV-VIII.	100 facts.	Per cents of error.	Reported data incomplete.
Wheeler (27)	1941	Controlled learning; game of MULT-O.	342—IIIB.	99 facts.†	Per cents of mastery.	Ranking of 99 facts† separately.

*The 78 "fundamental facts" are the twelve doubles 1 × 1 to 12 × 12 plus the sixty-six facts obtained when the direct and reverse combinations are treated together as single combinations, 1 × 2 (2 × 1) to 11 × 12 (12 × 11).

†MULT-O did not contain 0 × 0.

In order to permit ready comparison, the main facts with regard to the experimental procedures employed in the different original studies are assembled in Chart I. (Because several of the reports contain data from two or more separate researches it has been necessary to designate these sub-studies by letter, e.g., **10a**, **10b**, etc.) In Chart I the investigations are listed chronologically, are classified according to type of technique used, and are analyzed with respect to subjects, extent of combinations investigated, and the nature of the rankings for difficulty.

Investigations are classified according to type of technique as "controlled learning," "uncontrolled learning," and "status." Studies fall in the category "controlled learning" when methods of teaching were prescribed and supervised. In these studies observations, including test results, were obtained during the period of instruction or afterward or at both times. Such studies, of which there have been only five, vary considerably in the degree to which control was exercised over the teaching, but in every instance they employed subjects who were just beginning to learn the combinations. Customarily these studies were made in Grade III. Studies are classified as "uncontrolled learning" (a) when no attempt seems to have been made to prescribe and direct instruction (as a matter of fact, in such studies data have commonly been obtained from several schools which probably had quite different instructional programs) and (b) when measures were restricted to the results of tests given at the end of the period of instruction, usually at the end of Grade III. "Status studies" are those in which measures of knowledge were secured at points in time distant a year or more from the conclusion of instruction on the facts.

From Chart I several generalizations may be drawn:

(1) In only four investigations (**10a, 8, 14, 27**) were measures of learning difficulty obtained during the course of learning. In three other studies (**10c, 25b, 13**) measures of knowledge were secured at the immediate conclusion of the learning period. The rankings in the other studies are made from measures in which unknown and varied kinds of teaching, unknown degrees of mastery, and unknown amounts of drill and equally unknown ways of distributed practice had played their parts.

(2) Subjects have ranged all the way from children in Grade II (**10a**) to college students and college graduates, one of them with a Ph.D. degree (**2**).

(3) All except two investigators (**10, 27**) included the 100 separate combinations in their experiments—the 0-facts and the direct and reverse forms for each combination. One of these two investigators (**10**) did in a sense include the 100 basic combinations, but treated direct and reverse orders together; the other (**27**) omitted 0 × 0.

(4) A variety of criteria have been employed as measures of difficulty. The commonest has been the number or the per cent of errors, but this criterion has not always been used in the same way. For example, Holloway (**10a**) derived a coefficient of difficulty to allow for improvement, and also ranked the combinations in terms of their relative susceptibility to forgetting after a summer's vacation. On the other hand, Norem and Knight (**14**) used number of errors as only one of the three measures combined in their index of difficulty. Besides the criterion of number (per cent) of errors, other criteria which have been employed are number of practices in learning and in maintenance testing (**14**) and measures of time to supply products. Whether time is taken as the sole criterion or is combined with other criteria, the hypothesis is that those combinations are hardest which require the longest time to "think" the products. Even among studies employing the criterion of time, as shown in Chart I, the method for securing time measures has differed from study to study.

(5) The researches vary considerably also in complexity. One of the earliest studies, that by Holloway (**10**), remains as one of the most stimulating and comprehensive, for it explored difficulty from a number of angles. The Norem and Knight research (**14**) also rates high in complexity, five kinds of data on difficulty having been collected. At the opposite extreme in complexity are such studies as those by ReBarker (**16**), who added to the familiar Clapp technique of error comparison a different statistical treatment (sigma units), and by Smith and Eaton (**20**), who employed special apparatus to control the presentation of the stimulus cards in their group testing.

Limitations of space prevent a detailed criticism of the sixteen original studies. Instead, first, the conflicting results obtained in the studies will be illustrated; next, the reasons for the discrepancies will be considered; then, certain factors found to influence difficulty will be examined; and finally, all the research in this area will be evaluated with respect to its significance for the problem of instruction.

Conflicting Results

Chart II has been made up from data contained in eleven of the thirteen most recent research reports. The Holloway report (**10**) had to be omitted because therein difficulty rankings are given for each pair of facts (e.g., for 2 × 3 and 3 × 2 together) instead of for the separate combinations; and the data in the Washburn report (**23**) were not presented in a way which made comparisons possible. The chart contains columns for twelve studies, however, since the two investigations made by Washburne and Vogel (**25**) can be treated separately. Seven of the twelve studies are "status" studies, one an "uncontrolled learning" study, and four "controlled learning" studies.

In the first column at the left are listed the combinations which *in one research or another* were found to be among the most difficult five. The first five combinations, 9 × 7 to and including 9 × 8, are the most difficult five according to Smith (**19**). The second set of five, 7 × 0 to 4 × 0, are the most difficult five according to Clapp (**5, 6**). Batson and Combellich (**2**) report 9 × 7 and 6 × 9 (from the Smith five) as among their most difficult five, 96th and 100th, respectively; and the three combinations 4 × 4, 9 × 0, and 0 × 6 complete their most difficult five. Washburne and Vogel (**25**) rank only the most difficult twenty-five and the least difficult twenty-five; hence, all the combinations not in their first and fourth quarters are marked "M" in the chart. The most difficult five combinations in the other studies can be readily located.

If one reads the first line or row of Chart II one discovers that 9 × 7, which ranked 100 (most difficult) in the Smith investigation, was ranked 84 by Clapp, 96 by Batson and Combellich, "M" and 97 by Washburne and Vogel, 88.5 by Smith and Eaton, 81 by Anspaugh, 82 by ReBarker, 75 by Murray, 88 by Fowlkes, 89 by Norem and Knight, and 95 by Wheeler. The gross variation is from rank 75 to rank 100. The specific rankings assigned the other combinations, and the range in rank, can be determined from Chart II in similar fashion.

The first fact of significance yielded by a study of Chart II is that it takes a total of twenty-seven of the one hundred combinations to include all the combinations which have been ranked among the most difficult five (ranks 96 to 100) by some investigator. In all probability similar diversity would be revealed by a study of the least difficult combinations.

Second, wide range prevails generally in the ranks assigned each combination, as revealed in the last column but one. The smallest

CHART II

RANKS ASSIGNED "DIFFICULT" COMBINATIONS IN ELEVEN DIFFERENT STUDIES

Combinations	TYPE OF STUDY												*Range in Ranks*†	*No. of Studies in Most Difficult 25*
	Status							*Uncontrolled Learning*	*Controlled Learning*					
	Smith (19) (*Time*)	*Clapp* (5, 6) (*Errors*)	*Batson and Combellich* (2) (*Time*)	*Washburne and Vogel* (25) (*Errors*)	*Smith and Eaton* (20) (*Errors*)	*Anspaugh* (1) (*Errors*)	*ReBarker* (16) (*Errors*)	*Murray** (13) (*Errors*)	*Fowlkes* (8) (*Errors*)	*Norem and Knight* (14) (*Composite*)	*Washburne and Vogel* (25) (*Errors*)	*Wheeler* (27) (*Mastery*)		
9 × 7.......	100	84	96	M‡	88.5	81	82	75	88	89	97	95	75–100	10
6 × 9.......	99	79	100	M	91	37	78	73	100	100	79	97	37–100	9
9 × 6.......	98	80	74	78	91	62	74	65	96	85	M	98	62– 98	7
7 × 9.......	97	78	66	M	85.5	71	81	79	94	94	99	99	66– 99	9
9 × 8.......	96	64	36	M	69	39	71	56	82	62	M	79	36– 96	3
7 × 0.......	28	100	61	96	85.5	99	89	100	53	21	86	4	4–100	7
0 × 5.......	17	99	43	84	93.5	85	92	94	23	25	M	16.5	16.5– 99	6
0 × 7.......	19	98	31	89	99.5	91	94	93	14	42	80	36	14– 99.5	7
0 × 1.......	12	97	55	97	93.5	90	96	98	18	54	78	2	2– 98	7
4 × 0.......	30	96	90	98	85.5	97	86	99	61	11	93	18	11– 99	8
4 × 4.......	57	22	99	M	40	80	54	49	76	70	M	51	22– 99	3
9 × 0.......	11	90	98	M	50	100	85	83	60	3	97	7	3–100	6
0 × 6.......	21	91	97	83	97.5	83	98	87	21	44	M	26	21– 98	7
5 × 0.......	13	83	21	100	47	93	90	85	62	14	88	14.5	13–100	6
1 × 0.......	26	82	10	99	83	92	91	97	56	34	89	1	1– 97	7
8 × 0.......	9	89	15	97	49	98	84	86	58	33	85	28	9– 98	6
0 × 8.......	42	95	27	86	99.5	88	97	92	9	13	M	9	9– 99.5	6
0 × 4.......	15	93	22	85	97.5	84	93	96	30	46	M	13	13– 97.5	6
6 × 0.......	32	87	17	92	85.5	96	83	84	51	51	81	11	11– 96	7
0 × 9.......	27	86	59	91	95.5	86	100	91	15	29	83	27	15–100	7
0 × 2.......	36	94	6	87	95.5	89	99	88	22	30	76	20	6– 99	7
7 × 8.......	93	77	81	82	82	67	77	78	99	99	100	94	67– 99	11
8 × 7.......	94	81	95	81	78	74	76	58	98	83	98	92.5	58– 98	10
6 × 8.......	90	76	71	77	75.5	51	68	72	97	77	96	91	51– 97	7
7 × 6.......	72	68	38	M	75.5	58	64	60	80	98	84	96	38– 98	5
8 × 6.......	77	66	94	78	75.5	73	63	55	68	97	94	90	55– 97	7
4 × 8.......	68	60	37	M	70	52	56	63	70	96	17	81.5	37– 96	2
No. in most difficult 25..	9	22	9	19	18	17	20	17	10	10	19	11	......	......

*Murray's study was made with Scottish children who had just completed learning the combinations and averaged nine years in age.

†Exclusive of the two Washburne and Vogel studies.

‡"M" means that the combination fell in the middle half in difficulty, that is, that it ranked harder than the 25th in difficulty and easier than the 75th. Washburne and Vogel do not assign ranks to the middle fifty combinations.

range is twenty-five places, for 9 × 7; the largest is ninety-seven places, for 9 × 0. For the twenty-seven combinations the median range is eighty places out of a possible ninety-nine (rank 100—rank 1).

Third, the disagreement in rankings is especially serious in the case of the 0-combinations. The smallest range among the fifteen 0-combinations in Chart II is seventy-seven places; the largest, ninety-seven; the median, eighty-eight. Still, the median range for the twelve other combinations is forty-four places, and this is nearly half the possible range.

Fourth, on the average (median) a combination ranked among the most difficult five in one study is ranked among the most difficult *twenty-five* in but six other studies as well. Only one combination, 7 × 8, is ranked in the most difficult quarter in eleven of the twelve studies, and only two others, 9 × 7 and 8 × 7, in as many as ten studies. (The facts upon which these statements are based are in the last column of Chart II.)

Fifth, the twenty-seven combinations rank from ninth to one hundredth in the Smith study, from twenty-second to one hundredth in the Clapp study, and are about as widely distributed in the other studies.

Sixth, of the twenty-seven combinations listed in the chart only nine appear in the Smith and in the Batson and Combellich quarters of greatest difficulty. The median for the twelve studies is seventeen.

These brief summary statements are sufficient to indicate that the difficulty locations of the multiplication combinations differ greatly from study to study. Faced by the array of data in Chart II, one would indeed be in a quandary if asked to list the most difficult combinations in order. Which study should one adopt? Or, if one refuses to accept any one study, which grouping of studies should one accept, and why? Even the three "controlled learning" studies for which full data are available (**8, 14, 27**) rank the twenty-seven combinations differently by a median of twenty places, and rank ten combinations differently by at least twenty-five places.[10]

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[10] Murray (**13**) sees more uniformity among the findings of the various investigations than has been revealed in the present analysis. Thus, (**13**: 106-107) he points out that of his twenty-seven most difficult combinations, the 0-combinations being excluded, twenty are similarly regarded as very difficult by Smith, twenty-one by Clapp and by Batson and Combellich, nineteen by Fowlkes, and twenty by Norem and Knight. He finds similar agreement among the easy combinations, the 0-combinations being again excluded. On the other hand, the following facts should be noted: (a) Murray restricts his comparisons to five studies besides his own, thus leaving out of consideration

Reasons for Discrepancies

It is not at all surprising, after all, that the results of the studies summarized should differ so much. On the contrary, when one realizes the importance of conditions which varied from study to study, one could hardly anticipate more agreement than has been reported. Some of the sources for the different results may be briefly considered.

First, the process of ranking is itself a very crude procedure. At times it exaggerates small absolute differences in the criterion measure, and at times it minimizes large differences. Both tendencies may be illustrated from the Norem and Knight study (**14**), in which the criterion scale was unusually extended and in which therefore precise rankings might seem to have been easily possible.

The Norem and Knight scale, based upon three measures of difficulty which were combined by an undescribed method, extended from 456 points to 18, with a range of over 400 points. The five most difficult combinations had scores of 456, 435, 429, 419, and 412. Here a difference of one place in ranking is roughly equivalent to about ten points. On the other hand, ranks 81 to 77 (five combinations) are derived from scores lying between 207 and 200, and ranks 48 to 1 from scores between 49 and 18. In many instances two facts had the same composite scores and were assigned different ranks in an arbitrary manner. It is only fair to state that Norem and Knight recognized the unreliability of ranks in the middle of their distribution; yet they published the ranks of the one hundred combinations as they were obtained. Washburne and Vogel (**25**), on the other

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the results of another five studies which would have greatly decreased the appearance of agreement. (b) There seems to be little to justify the omission of the 0-combinations; after all they, like the other combinations, must be taught. (c) Furthermore, the elimination of the nineteen 0-combinations reduces the possible range in displacement from 99 to 80 and thus spuriously raises the appearance of agreement. (d) Only eleven of Murray's most difficult twenty-seven combinations are similarly placed in all of the other five studies, and only nine more are similarly placed in four studies. (e) Only nine of Murray's easiest twenty-seven occur among the easiest in the five other lists, and another eight in four lists. These facts do not seem to argue very strongly for agreement in research findings.

Wheeler (**27**) likewise argues for uniformity among the results of the different studies. His rankings correlated on the average .81 with nine other rankings, but four of the coefficients are .34 and below. When the 0-combinations are eliminated in the comparisons (seven this time), the median coefficient is .84 and all except one of the coefficient are above .80. (The exception is Batson and Combellich **2**, with a coefficient of –.35.) Had other available listings been included, for example, those of Murray (**13**) and of Smith and Eaton (**20**), the range in coefficients would probably have been increased, and greater variability revealed.

hand, reported the relative difficulty of only those combinations which lay in the hardest and the easiest quarters. They are the only investigators who have followed this practice.

Second, it is hardly reasonable to expect that "beginners" and "experts" should find difficulty at the same points. That is, difficulty may occur at one place in the initial stages of learning and at quite a different place at later stages. On this account, the discrepancies as between learning and status studies are understandable. Still, if this source of difficulty is removed and only studies employing the same time for measurement (i.e., near initial learning, or after the completion of learning) are taken for consideration, discrepancies remain, for these studies are subject to the influence of other uncontrolled conditions.

Third, the unlike criteria according to which difficulty has been assessed (indeed, even the unlike ways in which the same criterion has been applied) should cause differences in ranking. "Time to think the product" has yet to be shown equivalent to "number of errors in learning" as a criterion of difficulty. As a matter of fact, there is little possibility that these two criteria are interchangeable. One of the first of the difficulty studies (**10**) pointed out the nonequivalence of difficulty criteria, and hence the dependence of any list of rankings upon the criterion employed. In spite of this warning subsequent investigators have published their rankings as if they were unconditional.

Fourth, minor factors which may well have operated at the time the criterion measures were obtained have seldom been recognized or, if recognized, have seldom been evaluated as affecting the ranks assigned. Washburne and Vogel (**25**) showed that such a simple matter as location of a combination on the test sheet, at the beginning or at the end, affected the number of errors made and consequently was partly determinative of its ranking for difficulty. It is questionable whether exactly uniform conditions prevailed from combination to combination when the combinations were presented orally in the Re-Barker study (**16**) or were presented by means of apparatus in the Smith and Eaton study (**20**). Still again, one may doubt whether measures of time for "thinking answers" were directly comparable, subject to subject and combination to combination.

Fifth among the experimental conditions which have influenced placement for difficulty is the order and organization of the facts for learning. The combination 3 × 4 may offer one degree of difficulty

(a) if it is taught with 4 × 3, 2 × 4, 3 × 5, etc., or in a table, and quite a different degree of difficulty (b) if it is taught without temporal contiguity to these combinations and perhaps together with 2 × 9, 7 × 6, 1 × 8, and 4 × 5. In the former case (a), the child is aided by relationships which he can readily discover and use; in the latter case (b), he is virtually forced to memorize, and the results of memorization are impermanent. In every "controlled learning" study the investigator has had to face the problem of ordering and organizing the combinations for presentation to learners—has had to decide, for example, whether to arrange related combinations in tables. Even in "status" studies the number of errors for the various facts could scarcely fail to reflect the particular order and organization of combinations in initial learning. Regardless of type of technique used in an investigation, therefore, order and organization have had their effect on the difficulty measures collected. Yet it is rare to find in research reports any recognition of the extent to which difficulty rankings may have depended upon peculiar conditions in this respect. It is rare, too, to find any mention of the particular organization which characterized the teaching program—if, indeed, this was known.

The sixth experimental condition which affects learning difficulty is method of teaching, defined so as to include aspects of the instructional program not already mentioned. (This is the last condition to be considered here, though there are others.) Washburne and Vogel (**25**) noted great differences in the placement of the 0-combinations in their two studies. In their "controlled learning" study the 0-combinations were taught by means of principle (that is, with attention to meaning); and in this study the 0-combinations were found not to be particularly difficult. In their "status" study the 0-combinations had not been so taught, and their placement was at the opposite end of the scale of difficulty. Thiele (**21**) reports that when generalizations and understandings and "binding principles" were emphasized in learning, the 9-combinations were found to be next easiest after the 5-combinations, whereas they have commonly been reported as among the most difficult.

In spite of the many reasons to expect difficulty of the combinations to vary with methods of teaching, few reports have much to say about this matter. In the "status" studies nothing at all is said, though it is a fair guess that great variety of methods prevailed and that the unique features of the different programs represented in the co-operating schools disappeared in the general "average." Not much information as to instruction is given even in the learning studies. Holloway (**10**) states that two objective presentations for each combination were given "if time permitted," but he was clearly dubious about emphasis upon meaning and understanding. Fowlkes (**8**) had his subjects derive the products in each table by addition and thereafter relied upon drill to secure mastery. Norem and Knight (**14**) likewise did little to insure meaning for what was learned. They state that the combinations were first presented through addition, but that the tables were not taught as such; instead, the facts were taught in random order, save only that the smaller combinations were presented at the outset.

Factors Affecting Difficulty of Particular Combinations

Through various kinds of statistical analysis several investigators have sought to identify factors which tend to make combinations easy or difficult. In the following summary particular attention is paid to references **13** and **27,** in which Murray and Wheeler, respectively, report the most recent and thorough studies of difficulty.

1. The presence of 0 in a combination has commonly been found to make for difficulty. This fact can be readily identified in Chart II. That the difficulty of the 0-combinations is not necessarily excessive has been demonstrated in other studies (e.g., **25, 27**).

2. Difficulty of combinations, as measured by errors, and size of products are positively correlated. Murray (**13**) reports a coefficient of .78, which becomes .83 if the one atypical combination 1 × 1 (ranked 77) is eliminated. Coefficients for these two factors vary between .50, reported by ReBarker (**16**), when the 0-combinations were omitted, and .92 by Wheeler (**27**). Yet, Thiele (**21**) in his investigation found the 9-combinations, which yield some of the largest products, to be among the easiest for his subjects.

3. There is a positive relationship between the direct and reverse statements of the combinations. Murray (**13**) reports a coefficient of .34, so low that "a 'dynamic' relationship cannot be inferred" between associated pairs of combinations, but high enough to disprove an "antagonistic" relationship. Other reported coefficients lie between .21 (reported by Norem and Knight, **14,** as .76, but recalculated by Murray) and .80 (Wheeler, **27**).

4. Combinations involving the same pair of factors vary in difficulty. Thus, 7 × 9 may be easier or harder than 9 × 7. Murray

(**13**), Smith (**19**), and Clapp (**6**) find that the forms with the smaller factors first as multipliers are usually easier than the other members of the pairs; but this conclusion is challenged by others (Batson and Combellich, **2**; Fowlkes, **8**; Norem and Knight, **14**).

It is impossible to say to what extent these findings, even when they are generally supported by research, are valid for all conditions of teaching and learning. The present writers can imagine that the learning situation might be manipulated so as to yield data inconsistent with all four of the conclusions above. The reader is again referred to the Thiele study (**21**), in which most unusual findings were obtained when the instructional program was greatly modified from traditional practice. The reader is also referred again to the change which occurred in the placement of the 0-combinations under markedly different conditions of learning in the Washburne and Vogel research (**25**).

Evaluation of the Research for Educational Purposes

A disproportionate amount of space in this chapter has been devoted to the relative difficulty of the combinations. The only justification is that a corresponding disproportionate amount of attention has been given this problem in educational research. It has already been pointed out that half of the references in the chapter bibliography deal exclusively or primarily with difficulty rankings of the combinations.

The word "disproportionate" in the above statements implies a judgment of values. The space in this monograph and the amount of research on the relative difficulty of the combinations would not be disproportionate (a) if the research findings had established important new bases for instruction and (b) if the problem of relative difficulty were crucial to instruction. Condition (a) is not met because research has not resulted in a reliable and valid body of data. Condition (b) is not met for reasons which will be considered shortly.

With regard to condition (a) the rankings of two independent studies usually correlate .75 or less, and no one study or type of study seems to possess all of the merits and none of the weaknesses noted in the foregoing paragraphs. Ruch (**18**) has argued for the superiority of learning studies in general over what have here been called status studies, and has argued also for the Norem and Knight research (**14**) in particular, as compared with the research of Clapp

(5, 6) and Fowlkes (8). Yet, the Norem and Knight investigation has too many limitations to warrant the acceptance of the data obtained therein as being definitive, final, and universally valid.

Nor is the situation improved to any extent when an "average" or some other kind of compromise is sought. This procedure is illustrated in the work of Rock and Foran (17), who prepared two composite rankings from data reported by others. "Composite I" is a ranking for learning difficulty and is derived from Clapp's error data in Grades III and IV (6) and from the corresponding Grade III data published by Washburne and Vogel (25) and by Fowlkes (8). "Composite II" is for grades after the completion of learning and is based upon data from seven studies made above Grade IV. In both composites rankings were obtained by locating each combination in its *quarter* of difficulty (most difficult twenty-five, next most difficult twenty-five, and so on) and then by taking the means of these quarter-placements. The rankings of Composites I and II do not of course agree. How far this process of averaging results can eliminate weaknesses in the separate studies is an open question.

But the writers' criticisms go much beyond the technical limitations of the various studies. They are concerned with the significance of the problem itself, with the feasibility and usefulness of determining the relative difficulty of the combinations. In this connection three points will be made.

In the first place, the research on relative difficulty has involved the assumption that there is a final, fixed difficulty ranking for each combination, a difficulty which is unaffected by variations of any kind in the circumstances of learning. Accordingly, investigators seem to have set themselves the task of ascertaining *the* relative difficulty of the combinations, and persons of a theoretical turn of mind have spoken about differences in the "intrinsic" difficulty of the combinations. Now, in the writers' opinion there is *no* final order of difficulty, apart from the learner and all the conditions which govern his behavior. If difficulty is not ultimate and absolute and final and "intrinsic," then the most that an investigation can do is to find the difficulty order of the combinations for a particular set of circumstances. Here again one of the first investigators of this problem repeatedly warned his reader that his results held only for the study which he made. One quotation will serve to illustrate Holloway's caution, a caution which has not been at all generally heeded:

". . . difficult for whom? at what age and grade, and under what method of instruction?" (**10**:36).

The second point is that rankings have regularly been obtained by testing children who have "mastered" the combinations through repetitive practice, in accordance with a drill conception of arithmetic. If as the final test of knowledge one accepts glibness of answer (quickness, with accuracy but without understanding), then perhaps the studies of relative difficulty have furnished data of some value, however ambiguous they may be. If, however, one insists upon more than glibness on the part of the learner—in a word, if one insists upon understanding and the ability to use what is learned, then one cannot be content with difficulty rankings arrived at under conditions of instruction which are designed to attain other ends. As a matter of fact, the outcomes of meaning, understanding, and use can probably be achieved through a variety of programs. And if this is so, then a study based on any one program which stresses meaning and significance would be subject to the limitation mentioned in the immediately preceding paragraph: it could reveal the comparative difficulty of the facts only for that particular program.

The third point has to do with the frequently mentioned but seldom respected fact of individual differences. Consider the following cases selected from data reported by Norem and Knight (**14**):

Child	CA	IQ	Total Combinations Passed on Pretest	Total Learning Responses
A	7-11	146	48	532
B	9-6	103	5	2,565
F	8-11	126	65	516
L	9-0	126	17	3,271

One child (F) "knew" thirteen times as many combinations at the start of the experiment as did another (B) in the same grade. The youngest of the four subjects (A) "knew" nine times as many as the oldest. Subjects F and L had the same IQ, but L required six times as many learning responses. Differences of this kind are the rule among the children of typical classes. Can anyone suppose in the face of such facts that a single order of difficulty characterized these four children—or that practice based upon *any* order of difficulty would have been equally effective for all four? Is it not more probable that differences as to difficulty among these children varied as much from child to child as did CA and IQ (to mention but two important factors)?

Or, consider the case of a fourth-grade child found in the investigation to be reported in Part II. This child utterly failed to respect the findings of research as to combination difficulty. He had for some reason early come to useful knowledge of 7 × 9, rated generally as very difficult (see Chart II), and made use of this "hard" combination to find products for many others rated as much "easier."

Individual differences are important not alone in the academic laboratory of the psychologist; they are important in the classroom as well; and difficulty rankings have the deficiency of all averages, for they obliterate individual differences. It is the wise teacher who recognizes the significance of individual differences in general, in the learning of the multiplication combinations as well as elsewhere. And it is the wise teacher who provides learning experiences as needed by individual children regardless of the resemblance of her practice to the "facts" found by research on the difficulty of the multiplication combinations.

CONCLUDING STATEMENT

In this chapter the twenty-nine published reports of research have been reviewed under seven topics and sub-topics. For a statement of the findings the reader is referred to the closing paragraphs of the three main sections of the chapter. One of the main conclusions to be drawn from the whole body of research is that as yet we know exceedingly little about the teaching of the multiplication combinations. (With this conclusion Wheeler, **27**, would disagree.) Perhaps the chief reason for our lack of knowledge is that research studies tell us next to nothing about the way children actually learn the combinations. We know that learning is transferred (**4**), but we do not know what is transferred, nor how the transfer takes place. We know that drill develops proficiency in producing quick and accurate responses (e.g., **8**, **14**, **27**), but we do not know why, nor whether the results of repetitive practice are permanent and functional. We know that when taught in certain ways, some of the combinations are difficult and some are easy, but we can only surmise with regard to the reasons.

Only now and then in research reports does one encounter data on the process of learning; that is, data which tell us *how* children learn the combinations. The Thiele and Wheat investigations (**21**, **26**) show that meaningful experiences which precede abstract drill pay good dividends. Norem and Knight (**14**: 568) have shown that under conditions of drill children learn what numbers are products (e.g., 6, 10, 16, 18, etc.) and what numbers are not products (e.g., 17, 23, 31, etc.); that erroneous answers are generally the correct products for other combinations (91 per cent in their study) and are usually the products for closely related combinations (67 per cent); that when a combination becomes susceptible to error, other combinations closely allied to it also become susceptible to error. But, after all, these data are derived from a program of instruction which presented the combinations in random order with slight attention to understanding.

And we know also that for some reason the pairs of combinations (direct and reverse forms) are related by learners, that in general, but not consistently, children succeed better in learning combinations in which the first factor is the smaller, and, again in general but not consistently, difficulty mounts with the size of the product. But here, too, observations have been incomplete, for they have not revealed the reasons for the behavior which has been noted.

From what little has been reported in published studies—and just as truly from what has not been reported—it is possible to construct a description of the learning process which is rather generally supposed to hold in learning the combinations (a description which, however, would not be approved by Wheat and Thiele, among others). One shows the child—or lets him find out from one or two experiences—that 3 × 4 is the same as 4 + 4 + 4, and that the answers for both numerical expressions are the same; then through repetitive practice of various kinds one has the learner master the formula, "3 × 4 = 12." The most extreme statement of this conception of learning is found in Wheeler (**27**). (See Chapter X of this monograph.)

The limited success obtained by Wheeler (see p. 15, above) in teaching the multiplication combinations challenges, if it does not refute, the validity of this conception of learning. As a matter of fact, Wheeler concedes that his subjects made profitable use of certain generalizations, some of which they evidently discovered in spite of the method of instruction to which they were subjected; but his assumption that his subjects learned the other combinations by methods which corresponded to his instructional methods is wholly inferential and, in the opinion of the present writers, quite invalid.

In summary, then, the fact remains that our knowledge with respect to children's procedures in learning the multiplication combinations is meager. It is the purpose of Part II to present new data in this connection, with the further purpose of leading eventually to an improved methodology of instruction.

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Part II

Learning the Multiplication Combinations: Two New Studies

CHAPTER III

DESCRIPTION OF PROCEDURE

Part II of this monograph reports two studies in which the experimental procedures were very closely similar. Two techniques were used, namely, (1) group testing and (2) individual testing, or interviewing. The group tests consisted of the eighty-one basic multiplication combinations with the nineteen 0-combinations omitted; the combinations in the test were mimeographed in nine rows, nine combinations to the row. These group tests were given to all pupils in Grades III to V in the co-operating schools and were intended to yield the two commonly secured measures of mastery, (a) rate of work and (b) accuracy of answers. Individual interviews were then held with as many as possible of the children who had taken the group tests, the purpose being to obtain a third measure of learning, (c) the terms in which children thought about the combinations. This third measure is not usually obtained in educational research in the field of arithmetic.

The two studies were planned to discover as much as possible about the way in which children learn the combinations by securing a series of five cross-section descriptions, as it were, at as many spaced intervals throughout two and a half years of schooling, from the end of the third grade to the end of the fifth. Group tests and interviews were accordingly scheduled for the first week in December in Grades IVB and VB and for the first week in May in Grades IIIA, IVA, and VA. While it is common practice to expect mastery of all combinations by the end of Grade IVA, tests and interviews were given in Grades VB and VA also, so as to picture learning at its latest as well as at early and intermediate stages.

The first study, reported in Chapter IV, involved four schools, two in Burlington and two in Raleigh, North Carolina, and was conducted by the senior author and by thirteen of his graduate students at Duke University. All group tests and interviews in this study were administered by these persons after numerous conferences to work out methods of testing, interviewing, recording, and interpreting which could be held to consistently. Since this "intensive" study was made in near-by schools, it was possible to supervise the collection of data in a way which was impossible in the second and larger investigation. This second study, which is reported in Chapter V, involved schools in eight states; and local teachers, principals, and supervisors had to assume responsibility for gathering data. With these persons contact could be maintained only through correspondence.

DETAILS OF THE PROCEDURE

Subjects

The number of subjects in the two investigations totaled more than 4,000. Of these, 3,619 furnished records which could be used in one or more phases of the research. About 250 papers were lost from the first group testing because pupils failed to follow directions for keeping time records. Other losses occurred in the second group testing because of absences. Somewhat fewer than 1,900 children were interviewed, all of them once, and enough twice to yield a total of 2,887 interview records.

The distribution of subjects who supplied group test data is given in Table 1, by half-grades, states, and school systems. Corresponding facts for interview subjects are summarized in Table 2. In both tables a distinction is indicated between the groups of subjects in the local or intensive study (all from North Carolina) and those in the extended study (eight states, nineteen school systems). The average number of subjects per half-grade is about twenty-five, and both rural and urban communities are represented.

It has already been explained that contact with the schools in the larger study could be set up only by mail. In each co-operating system one teacher was appointed as co-ordinator, to receive and distribute supplies and instructions. In addition to personal letters to these co-ordinators, full mimeographed instructions and explanations were sent through them to all teachers. Four such communications were employed: (1) "General Instructions"; (2) "The Group Test"; (3) "The Individual Test"; and (4) an untitled letter reminding co-ordinators, under date of April 4, of the necessity for organizing for the spring testing and interviewing. All teachers in Grades IV and V received communications (1), (2), and (3) in the fall; teachers of Grade III received only communication (1) at that time, (2) and (3) being sent them in the spring. Copies of the first three communications will be found in the Appendix.

TABLE 1
NUMBER OF USABLE GROUP TEST PAPERS, BY HALF-GRADE AND BY SCHOOL SYSTEM

State	*School System*	*Number of Classes, by Half-Grades*	*Number of Usable Group Test Papers, by Half-Grades*				
			IIIA	*IVB*	*IVA*	*VB*	*VA*
Intensive study (Chapter IV)							
North Carolina	Burlington	2	(60)*	59	56	65	55
	Raleigh	2	(64)	58	52	62	62
Total		...	(124)	117	108	127	117
Extended study (Chapter V)							
Florida	Clearwater	1-2	(41)	50	41	55	51
	Cortez	1	(7)	7	6	5	5
	Lake Worth	1	(40)	29	24	44	43
Kentucky	Versailles	1-2	(25)	50	48	39	33
Mississippi	Gunnison	1	(26)	31	25	32	30
	Lumberton	1-2	15	19	16	17	14
	Nettleton	1	11	14	12	17	13
	New Albany	1	(19)	31	26	29	27
	Oxford	2	(54)	59	44	68	62
North Carolina	Raleigh	1	(17)	33	28	25	20
	Taylorsville	2	56	54	47	76	69
	Wake Forest	2	51	55	43	62	58
Pennsylvania	Nanty-Glo	1-2	(47)	44	43	74	69
South Carolina	Orangeburg	2	65	62	56	67	61
Virginia	Painter	1	(15)	11	12	11	11
	Parksley	1	(15)	20	16	24	24
	Temperanceville	1	(18)	18	14	17	13
West Virginia	Oakvale	1	(35)	26	22	35	33
Total		...	557†	613	523	697	636
Grand Totals		...	681	730	631	824	753

*The numbers in the parentheses stand for papers of pupils who had studied fewer than the 81 multiplication combinations in the test. In some cases the combinations had been studied through the tables of 5's; in others, through the tables of 6's or 7's.

†The total 557 included 359 children who had studied fewer than the 81 combinations and 198 who had completed the whole series of tables.

The Group Test

Page 172 of the Appendix contains a sample of the group test on the eighty-one basic combinations (the 0-combinations being excluded). This test was given once in Grade III, in May, and twice in Grades IV and V, in December and May, in order to secure measures of rate and of correctness of answers at the ends of Grade IIIA, IVB, IVA, VB, and VA. Although some of the Grade IIIA classes had studied fewer than the eighty-one combinations, they were never-

TABLE 2

NUMBER OF SUBJECTS FROM WHOM INTERVIEW DATA WERE OBTAINED, BY HALF-GRADE AND BY SCHOOL SYSTEM

State	*School System*	*Number of Classes, by Half-Grades*	*Number of Interviewees, by Half-Grades*				
			IIIA	*IVB*	*IVA*	*VB*	*VA*
Intensive study (Chapter IV)							
North Carolina	Burlington	2	(78)*	76	76	70	70
	Raleigh	2	(70)	60	60	68	68
Total		...	(148)	136	136	138	138
Extended study (Chapter V)							
Florida	Clearwater	1-2	(37)	46	38	19	18
	Cortez	1	(6)	9	9	5	5
	Lake Worth	1-2	(45)	29	26	24	20
Kentucky	Versailles	1-2	11	23	15	11	8
Mississippi	Gunnison	1	(18)	30	25	25	15
	Lumberton	1-2	21	17	17	16	16
	Nettleton	1	11	12	12	14	14
	New Albany	1	(15)	16	14	15	14
	Oxford	2	(54)	55	48	59	51
North Carolina	Raleigh	1	(13)	14	14	25	22
	Taylorsville	2	57	41	36	40	34
	Wake Forest	2	19	17	16	18	16
Pennsylvania	Nanty-Glo	1-2	(43)	48	47	48	47
South Carolina	Orangeburg	2	32	63	51	15	14
Virginia	Painter	1	(18)	14	13	13	13
	Parksley	1	(27)	10	10	28	26
	Temperanceville	1	(19)	10	9	16	16
West Virginia	Oakvale	1	(28)	29	25	35	24
Total		...	474	483	425	426	373
Grand Totals		...	622	619	561	564	511

*The numbers in the parentheses stand for children who had studied fewer than the 81 multiplication combinations in the test. In Chapters IV and V interview data are used only for the combinations which were supposed to have been mastered. Of the 622 third-grade children, 334 had studied fewer than the 81 combinations.

theless asked to take the whole test, in order to permit certain comparisons between responses to learned and to unlearned combinations.

The children were told to work as rapidly as possible on the group tests without making errors. If they encountered combinations which were too difficult for them, they were to omit these and to go on to the next better known combinations. In the scoring, one point was allowed for each correct answer. Thus, the highest score was 81 points.

The measures of rate of work were obtained in the following manner: The teacher (or examiner) announced before the test that he would say "Mark" several times during the testing. Each time they heard this word, the children were to underscore or otherwise indicate the combination on which they were at work, and to go on at once without stopping. It was emphasized several times that "Mark" did not mean to stop work. The pupils did not know how often this signal was to be given; as a matter of fact, it occurred each thirty seconds. No time limit was set for the test. In each class work was continued until all except the slowest three to five pupils had finished.

The "marks" were then used to determine rate of work. A count of the "marks" on each paper told how many full thirty-second intervals were used. To the total time represented in these units was added a proportionate time for the combinations at the end of the test after the last "mark."

This method of timing was not wholly satisfactory, for several children, particularly in the lower grades, neglected to make some of the marks when time signals were announced. When it was impossible to arrive at a fair estimate of the time for such a pupil by using such data as his paper afforded, the paper was rejected both for the study of rate and for the study of accuracy. Other papers were rejected when children skipped a whole row of examples. For these two reasons together, about 7 per cent of the papers were eliminated in Grade IIIA, 9 per cent in Grade IVB, and less than 5 per cent for the grades above that point. A negligible number of papers (1.5 per cent) were discarded because of illegible answers.

The Interview

To secure data with respect to thought processes, interviews were arranged with individual pupils within two weeks of the group tests. All children tested in the local study could be interviewed as well, for there were enough interviewers. This was not true in the extended study. In the latter case teachers were asked to interview all their pupils if they could, but, failing this, to select subjects in a manner which would guarantee a representative sample. They were to interview the first pupil on their alphabetical rolls, then the second, the third, and so on, going as far through the rolls as time permitted. Actually, comparison of the figures in Tables 1 and 2 will show that these teachers were exceedingly generous in their co-operation. When the reader realizes that each interview took fifteen minutes, he can appreciate the extent of the service rendered.

Fifteen combinations were used in the interviews. These combinations were chosen so that (1) all tables should be represented and (2) in general only the more difficult ones would be used. The selected combinations were then arranged in an order which minimized the possibility of using one combination to solve the next.[1] Thus, 5 × 7 was followed, not by 5 × 6, by 5 × 8, by 4 × 7, or 6 × 7, but instead 9 × 6 which contains neither factor in 5 × 7.

Prior to the start of the investigations, a preliminary study was made in order to secure categories for classifying pupils' thought processes in dealing with the combinations. In this study more than thirty pupils were interviewed, part by the writers and the rest by graduate students under their observation. A list of eleven categories are explained later in this chapter.

In the interviews children were of course taken one at a time. Cards had been prepared, each containing one combination. After explaining what was expected, the interviewer exposed a practice card for a combination not used in the research proper, asked the child to state his answer, and made careful observations of all relevant behavior. In many cases the child revealed his procedures by "thinking out loud," by lip movements or hand movements, or in some other overt manner. When this was not true, the interviewer undertook to learn from the child what processes he had used, carrying the questioning just far enough to get this information without supplying cues and without making him self-conscious of his processes. Work with the practice cards was discontinued only when the interviewer was satisfied with respect to *rapport* and to the child's understanding of his task.

Then the research interviewing began. The interviewer exposed the first combination card and noted on the record form (a copy is on page 51): (1) his numerical answer, whether correct or incorrect, (2) the code letter which corresponded to the appropriate category for his thought pocess, and, in certain cases when there was any doubt as to the pupil's procedures, (3) the actual words used by the child to describe what he had thought. These exact reports were then used by the staff at Duke University to classify, if possible, all doubtful responses.

On page 52 will be found a description of the categories for classifying children's procedures, together with the corresponding code letters. It will be noted that the code letters are repeated at the bottom of the record form (below), each with a brief definition, and that illustrative cases of interviewed subjects appear at the top of this same form.

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[1] The range of "difficulty" among the fifteen combinations is from rank 47 to rank 92 according to the Norem and Knight list, with the median rank 76. The range according to the Clapp list is 13 to 83, with the median at 58.

All categories except the first three proved to be entirely satisfactory. That there might be trouble at these three points was known before the local investigation was started,[2] and steps were taken to

INTERVIEW BLANK

Date____________________

School________________________ Town____________________ State____________

Teacher_______________________ Grade________ Interviewer____________________

Pupil's name	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7 6	9 2	4 7	5 3	8 6	7 9	3 6
Joe H.	35 H	N	24 R	30 H	56 T	24 H	21 H	72 X 65+8	42 T	18 H	28 H	15 H	48 T	52 T	18 H
Susie Q.	35 H	54 S	24 R	30 H	56 S	24 H	21 H	75 G	42 T	18 H	28 H	15 H	48 S	60 S	18 H

H – Habituation, meaningful
M – Memory, rote
G – Guessing
S – Solution

C – Counting
R – Reversal
T – Tables
V – Visualization

N – No Attempt
I – Indeterminate
X – Miscellaneous (describe)

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[2] In her Master's thesis, Miss Bertha Nichols was not always able to distinguish with certainty between the processes here designated as G, M, and H. Bertha Nichols, "How Children Learn the Multiplication Facts." Unpublished M.Ed. thesis, Duke University, 1934. 55 pp.

Code Letter	*Description of Class*
H	*Habituation, meaningful.* The correct answer is confidently given at once, *with every indication of understanding.*
M	*Memory, rote.* The answer is given at once and with apparent confidence, *but there is no evidence of understanding.*
G	*Guessing.* The answer, usually wrong, is given rather promptly, but it is evident that the child is guessing.
S	*Solution.* The child starts with a familiar combination, and adds to, or subtracts from, the product to get the answer. There are many kinds of solution, for example: $3 \times 4 = 2 \times 4 = 8, + 4 = 12$ $3 \times 4 = 4 \times 4 = 16, - 4 = 12.$
C	*Counting.* The child adds or counts *the same unit* several times. $3 \times 4 = 3, 6, 9, 12$; or, 4, 8, 12.
R	*Reversal.* The child interchanges multiplier and multiplicand, in order to get a more familiar order. $6 \times 3 = ?$; $3 \times 6 = 18$, so $6 \times 3 = 18$.
T	*Tables.* The child starts with a lower combination in the particular table involved and recites the table to the required point. $4 \times 6 = ?$; $4 \times 1 = 4$; $4 \times 2 = 8$; $4 \times 3 = 12, \ldots$ or, $4 \times 4 = 16$; $4 \times 5 = 20$; $4 \times 6 = 24$.
V	*Visualization.* The child reproduces groups of objects or figures in clear imagery and works with these. $4 \times 5 = ?$; $4 \times 5c$ (nickel) $= 20c$, or 20.
N	*No attempt.* The child makes no attempt to state the answer.
I	*Indeterminate.* The child gives an answer for the combination, but you cannot ascertain *how* he got it (that is, his processes).
X	*Miscellaneous.* The child reports a process not included in the preceding list—or you are uncertain as to its proper classification. Report what the child said, using the back of the Interview Blank if this is necessary.

minimize inaccuracies arising from this source. Indeed, it was this foreknowledge which inspired the preliminary interviewing with which the reader is already familiar (p. 50, above).

"Guessing" caused less trouble than did "rote memory" (M) and "meaningful habituation" (H). Incorrect answers that were meaningless, as shown by the child's inability to justify his answers, could be safely entered as "G." No such certainty attached to processes, possibly "Guessing," which led to correct answers. Quite probably, some entries under "M" and "H" belong more properly under "G," but the number of such errors is not large. In most instances the questioning which followed responses classified tentatively as G revealed the validity or the invalidity of this classification.

So far as H and M are concerned, the following procedure was adopted: (1) A tentative entry of M or H was recorded at once for combinations with which one of these processes seemed to have been used. At this time questioning was carried only far enough to give a reasonable basis for preliminary classification. Then after the subject had responded to all fifteen interview combinations, he was taken back over the M and H facts. (2) On the second trial the child was asked, "But how did you know that . . . was the correct answer?" and, "What did you think to get the answer?" If these questions did not elicit an unambiguous report, an absurd suggestion was given. For example, for 8 × 3 it might be asked, "Did you get your answer by thinking '8 and 3 are 11, and 7 are 18, and 6 are 24'?" In his eagerness to avoid suspicion of so ridiculous a method, the child usually revealed what he had thought. If this explanation warranted an unequivocal classification as to process, the next doubtful combination was presented. If not, the original entry of H or M was allowed to stand until the remaining doubtful facts had been considered. The child was then taken back over the uncertain M and H facts. (3) On the third trial the interviewer said, "But suppose I say that 24 (for 8 × 3) is not the correct answer. Suppose I say that the answer is 25. How could you prove me wrong?" If the child was able by any rational and logical method to prove the correctness of his answer and the incorrectness of the suggested answer, it was assumed that he understood what he was doing, and he was given credit for H. If, on the other hand, he had no way of testing the correctness of answers, his process was classified as M.

This whole procedure sounds much more complicated in the description than it was in actual use. The child was not aware of the three levels of questioning, and so his responses remained true to reality and unaffected by the inquiry. For this reason the figures given below for M and H in the local study are offered with considerable confidence.

As will appear in Chapter V, the data for G, M, and H in the extended study do not wholly agree with the corresponding data for the local study. The interviewers in the extended study were not furnished statements of the carefully worked out procedures (just described) for distinguishing among the three processes. These directions could be explained and illustrated simply enough when instructor and interviewer were face to face; but the prospect of making them clear by mail seemed very doubtful. On this account reliance had to be placed in statements much less specific. In this connection the relevant paragraphs from the "General Instructions" are quoted below. It may be added that later, in the communication "The Individual Test," teachers were advised to read again these cautionary paragraphs before attempting to master the specific instructions and explanations for interviewing.

The group tests, you will find, will be easy to administer, for you have had many experiences in giving similar tests. The individual tests, on the other hand, are not easy to give. On the surface, the interview appears to be a very simple procedure for securing research information; in reality it is a very difficult procedure to employ effectively. This is especially true in investigations such as this one, in which one seeks to get at children's mental processes and manipulations.

The interviewer should observe not only the subject's *verbal responses,* but also his *facial expressions, lip movements,* and *mechanical activities,* since all of these may reveal what the child is thinking. A normal, friendly relationship should exist between interviewer and subject. The interviewer must be careful, however, to *avoid approval or disapproval* and the giving of cues or suggestions. (The child is apt to adopt whatever suggestion is offered in place of his true process.) Also, the interviewer must exercise caution *not to push the questioning too far,* lest the child become self-conscious, and his responses and methods cease to be typical.

These warnings may seem to be unnecessary, but actually they are not. Teachers who have had the most experience with the interview will be the first to recognize the need of the cautions and to advise their less experienced friends to heed them. Every possible safeguard must be taken to guarantee *true records* of children's thought processes in dealing with the interview combinations. You will soon discover that an apparently "known" combination may not be "known" at all, but that the child has recited the tables, or guessed, or solved from better known combinations, or counted, or given a glib, meaningless answer, even if it be correct. It is precisely this kind of information which the interview is designed to supply. You see, for the purposes of this part of the investigation, the correctness or the incorrectness of the answers is relatively unimportant; what we want to know is how the child *thinks* to get his answer.

EVALUATION OF THE PROCEDURE

The techniques used in the present investigations provided three measures of learning: rate, accuracy, and type of thought process; and they provided these measures at five points in learning, each pair of which were separated by a half-grade of instruction, from Grade IIIA to Grade VA. A combination of the three measures would, it was hoped, shed considerable light on the course of learning the combinations from its start to a stage which may be called mastery. Each kind of measure contributes its share of illumination. The two measures, rate and accuracy, are the traditional measures both of research and of the classroom. They reveal the efficiency which the child has attained in dealing with the combinations, this and this only. To complete the picture of learning, it was necessary therefore to secure data of a different and of a more intimate kind; accordingly, the interview was employed in order to disclose what changes, if any, in level of quantitative thinking characterize progress toward mastery. Data for this purpose were not obtainable from test papers, for answers reveal little concerning the processes which lie behind them.

The interview is not exactly popular in educational research on learning. The reason lies in doubt concerning the reliability and perhaps the validity of the data secured thereby. In the present instance, as usual, there is no statistical evidence for the reliability of the interview data. Yet, the writers have confidence in their results with respect to thought processes, particularly in those obtained in the local study. The major source of error in these results has already been mentioned, namely, some uncertainty with regard to entries under "G," "M," and "H." But, as later chapters will show, errors of classification in these categories can affect very little the findings as a whole. The writers' confidence in their interview data arises from their extended experience in observing children busy with tasks which they have not yet completely habituated. In performing such tasks children many times reveal their procedures in their overt activity; at other times they can make their implicit procedures known by describing their thought processes in words. There is no way in which the writers can communicate their confidence to the reader, except possibly to urge him to check the data in the present studies by his own interviews, which should match those in the present studies in care, insight, and thoroughness.

Another critical phase of the technique relates to the degree to which a continuous picture of development (or a picture of continuous development) can be achieved by the research methods employed here. Ideally, one should follow the same group of children from grade to grade for two and a half years and should take frequent "soundings" during this period. Instead, these investigations made use of a less ideal, though much commoner, method. In place of one group, three different groups of children were observed; and in place of frequent observations, but five were made, and these at intervals of several months. In justification one can but state that the ideal procedure was impracticable—indeed, is practicable only to the teacher who accompanies his pupils through several grades—and that the procedure which was practicable held large promise of useful results.

CHAPTER IV

RESULTS OF THE LOCAL INVESTIGATION

The test and interview results obtained in the two Burlington and the two Raleigh, North Carolina, schools will be discussed in this chapter. The corresponding results for the extended investigation will be treated in a similar manner in Chapter V. The reasons for separating the two studies have already been made clear.

An average of nearly 140 children per grade participated in the local study. These pupils were drawn, in the case of Burlington, from one school above the city average in quality of achievement and from a second school below the city average by about the same amount. "Average" in both instances was determined by standings on the new Stanford Achievement Test, which had just been administered throughout the system. In Raleigh, on the other hand, the two cooperating schools were selected by the city elementary supervisor so as to secure a typical cross section of the city school population.

The data in Table 3 show the degree to which the pupils in the local study may be regarded as representative of their respective grades. The measures of MA and IQ were obtained in Burlington by means of the California Mental Maturity Scale and in Raleigh by means of the Kuhlman-Anderson Tests. The measures of Arithmetic Age (AA) were secured in both cities by the new Stanford Achievement Test, the scores on the computation and problem-solving sections being combined according to directions. The testing for intelligence and arithmetic achievement was done in Grade IIIA in the early part of May and in Grades IVB and VB in the early part of December—in all instances, then, at about the same time as the testing and interviewing on the multiplication combinations.

In IQ the children were very slightly above average, probably because all repeaters were eliminated in this study. Arithmetic achievement in all grades was at about the norm. The median AA in Grade IIIA (115 mos.) is equivalent to a grade standing of III:8; the median AA's in Grades IVB and VB (119 mos. and 134 mos.) are equivalent to grade standings of IV:1 and V:3, respectively.

Both school systems make use of the same state-adopted arithmetic textbook, and both follow the state course of study in arithmetic. In Raleigh the third-grade pupils, at the time of the testing, had

TABLE 3

DATA ON THE PUPILS TESTED AND INTERVIEWED IN THE LOCAL STUDY

Grade, and Measure	*Md*	Q_1	Q_3	*Range*
Grade IIIA:				
CA (mos.)	110	106	117	99 - 153
MA (mos.)	114	105	119	79 - 134
AA (mos.)	115	103	122	* - 129
IQ	103	90	112	61 - 129
Grade IVB:				
CA (mos.)	119	114	123	99 - 157
MA (mos.)	122	113	131	90 - 162
AA (mos.)	119	113	125	96 - 140
IQ	103	94	115	71 - 135
Grade VB:				
CA (mos.)	129	124	134	111 - 163
MA (mos.)	136	126	143	93 - 162
AA (mos.)	134	127	137	110 - 146
IQ	105	96	113	65 - 129

*The lowest scores made in the testing were below those for which AA's are assigned in the new Stanford Achievement Test.

just completed study of the multiplication tables for 5 and for 5's and were supposed to have mastered a total of sixty-five combinations. In Burlington study of the combinations, so far as mastery is concerned, had advanced to the same point, though the pupils had had "some preliminary experiences" with the tables for 6 and for 6's. In both systems the multiplication tables through those for 9 and for 9's are taught in Grade IVB and had been completed before the December testing. At the time of this study, drill (repetitive practice) of various kinds was the principal method for teaching the combinations. Only limited use was made of means for giving children an understanding of the process of multiplication and a grasp of the relationships among the combinations. Because of these similarities—similarities of content, of its organization, and of methods of teaching—it was felt that data from the two systems could be combined as if from a single program of instruction.

It is important both at this point and throughout the chapter to bear in mind that the data to be presented *are* the results of a single program of instruction. Obviously there are other programs, and these programs may differ materially in organization of the facts for learning, in methods of teaching, in the amount, kinds, and distribution of practice for learning, and so on. Obviously also these differing programs could easily produce results in accuracy, rate, and thought processes unlike those in this chapter. The reader will do well to insert the phrase "for this program of instruction" wherever in the chapter this limiting phrase (or one like it), omitted for the sake of brevity, may seem to be essential. Certainly the writers have been continuously aware that their findings are not of universal validity.

GROUP TEST DATA

Accuracy Scores

Table 4 contains the distribution of accuracy scores for all five half-grades, arranged in class intervals of two points. Thus, 13 of the 124 Grade IIIA subjects made scores of 80 or 81 (81 is the highest possible score), 3 made scores of 78 or 79, and so on. The range in scores for Grade IIIA is from 36 (see the footnote to the table) to 81, with the median (determined from the test papers) at 64 and Q_1 at 55. In view of the fact that these children were supposed to "know" only sixty-five combinations, the median of 64 is not low, but high. More than 40 per cent of the pupils made scores above 65, and six had perfect scores.

The distribution for Grade IVB reveals the expected improvement from Grade IIIA. The range is somewhat reduced, Q_1 is 75, and the median has been raised to 80 (nearly 99 per cent of perfect). Of the 62 who made scores of 80 or 81, 36 (31.9 per cent of the whole group) made perfect scores. Except for the wide range in achievement, the Grade IVB children seem to have been well on the way to consistently correct answers for the combinations.

The facts with regard to the accuracy scores for Grades IVA, VB, and VA are self-evident: in all three half-grades the median is 81, and Q_1 is 80. In other words, three fourths or more of the children in each of these three half-grades made scores of 80 or 81. The numbers actually making perfect scores were 64, 78, and 65 for Grades IVA, VB, and VA, respectively, and these numbers represent 59 per cent, 61 per cent, and 56 per cent of the populations. By the end of Grade IVA the eighty-one combinations seem to have been mastered, as far as ability to give correct answers is concerned. But mastery must be defined to include, besides (a) accuracy of answers, (b) promptness of answers, and (c) maturity of thought processes as well. The second aspect of mastery is considered next.

Rate Scores

Table 5 is similar to Table 4 in construction. It shows the distribution of rate scores in class intervals of a half minute. The data

TABLE 4

LOCAL STUDY: ACCURACY SCORES ON THE GROUP TEST

Score	*Frequencies, by Half-Grades*				
	*IIIA** (*N=124*)	*IVB* (*N=117*)	*IVA* (*N=108*)	*VB* (*N=127*)	*VA* (*N=117*)
80	13	62	92	101	96
78	3	18	8	14	14
76	6	7	6	7	4
74	3	4	1	1	1
72	14	5	1	..	1
70	7	7	..	1	..
68	9	2	..	2	..
66	7	1	..	1	1
64	7	4	..	..	..
62	8	2	..	..	..
60	5	..	..	..	..
58	5	..	..	..	..
56	9	..	..	..	..
54	4	1	..	..	..
52	5	2	..	..	..
50	5	1	..	..	..
48	6	1	..	..	..
47, and below	8†	..	..	..	..
Actual Md	64	80	81	81	81
Actual Q_1	55	75	80	80	80

*In Grade IIIA the pupils had studied only sixty-five of the eighty-one combinations, with the expectation of mastery.

†The eight scores are distributed as follows: 46 (1); 44 (1); 42 (1); 40 (2); 38 (2); 36 (1).

for Grade IIIA are read: 1 of the 124 Grade IIIA pupils finished the group tests in about two and a half minutes (interval, 2:30-2:59); none finished more quickly than this, and no others in fewer than three and a half minutes (4 finished in three and a half to four minutes), and so on.

Two facts may be noted about the distribution for Grade IIIA: (1) the general slowness of the group and (2) the range in time scores. Nearly one fourth of the children required at least eight minutes and a half, and half required six minutes, twenty seconds. Only a quarter finished in fewer than five and a half minutes. Still, these children were struggling with sixteen combinations which they had not studied. Unfortunately the time spent on the studied combinations cannot be separated from the time spent on the unfamiliar facts.

The records for Grade IVB show a reduction in median time of almost two minutes; there is the same reduction in Q_3 and even greater reduction in Q_1. When the grade median of four minutes,

TABLE 5

LOCAL STUDY: TIME RECORDS ON THE GROUP TEST

Time	*Frequencies, by Half-Grades* *IIIA* (*N=124*)	 *IVB* (*N=117*)	 *IVA* (*N=108*)	 *VB* (*N=127*)	 *VA* (*N=117*)
1:00..........	..	..	..	1	1
1:30..........	..	..	3	1	2
2:00..........	..	3	22	7	23
2:30..........	1	10	31	33	43
3:00..........	..	16	20	29	24
3:30..........	4	12	11	32	13
4:00..........	12	14	6	10	6
4:30..........	8	18	3	7	4
5:00..........	8	14	7	2	..
5:30..........	6	8	3	2	..
6:00..........	16	6	..	2	..
6:30..........	14	1	1	..	..
7:00..........	6	8	1	1	1
7:30..........	15	1	..	..	..
8:00..........	4	2	..	..	..
8:30..........	7	1	..	..	..
9:00..........	4	..	..	..	..
9:30..........	4	1	..	..	..
10:00..........	1	1	..	..	..
10:30..........	5	..	..	..	..
11:00, and over .	9*	1†	..	..	..
Actual Md.....	6:20	4:40	2:45	3:25	2:50
Actual Q_1......	8:25	5:30	3:50	3:50	3:25
Actual Q_3......	5:20	3:30	2:30	2:45	2:30

*The nine time records distribute as follows: 11:00 (1); 11:30 (3); 12:00 (2); 12:30 (1); 13:00 (1); 13:30 (1).

†Actually, 12:20.

forty seconds is compared with the medians for the subsequent grades, however, it is evident that much improvement is still to be made in promptness of answers.

Rate of work seems to be about as good in Grade IVA as it will be in either of the two later half-grades: the median is within five seconds of the Grade VA median (and is forty seconds better than the Grade VB median), and the Grade IVA Q_1 and Q_3 compare favorably with the corresponding measures for Grade VA. It is not far from the truth to say, then, that after Grade IVA little improvement takes place either in rate or in accuracy of work. Mastery when measured by these two indices is practically attained by the end of Grade IVA. Data with respect to the third criterion or aspect of mastery, namely, thought processes, will be treated shortly.

Accuracy and Rate Considered Together

One way to show the course of improvement in both rate and accuracy is by means of curves, as in Figure 1. To construct these curves the accuracy and rate records (medians and Q_1's) made in Grade VA were taken as bases, and percentage figures for the other half-grades were derived from these bases. For example, the accuracy median for Grade IIIA is 64; that for Grade VA is 81; and the ratio of the former to the latter is 79.0 per cent. The rate median for Grade IIIA is six minutes, twenty seconds, which becomes 44.7 per cent when compared with the Grade VA median of two minutes, fifty seconds. (This is to say that the average Grade IIIA child worked 44.7 per cent as rapidly as did the average Grade VA child.) Percentage expressions for Q_1's were computed in a similar manner.

The curves show again in more tangible manner the facts that have been drawn from the tables. In accuracy the children at the end

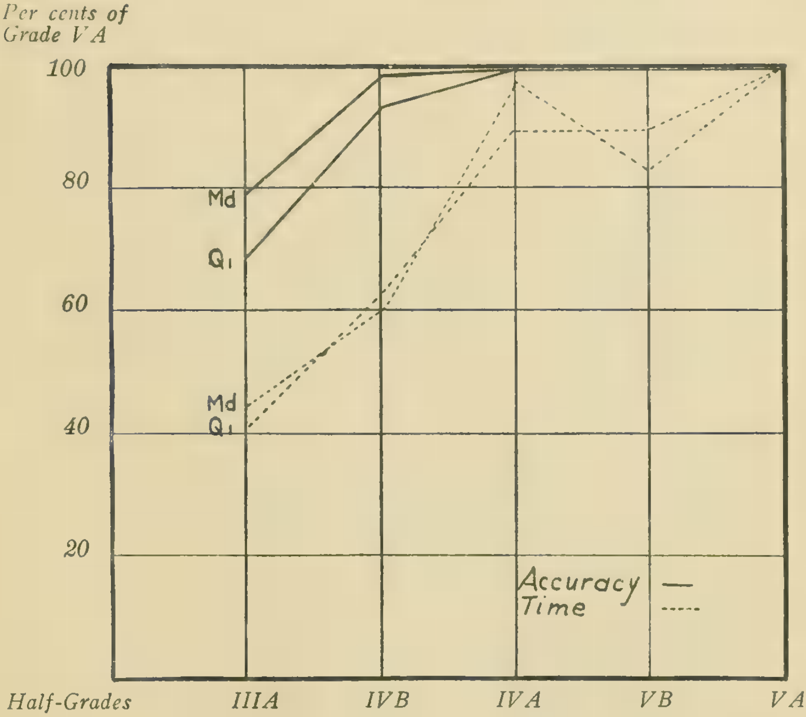

FIG. 1. Local study: increase in accuracy and rate by half-grades expressed in terms of the Grade VA medians and Q_1's.

of Grade IVA were about as far along as they would be a year later. This statement holds whether the comparisons are based upon medians or upon first quartiles. As a matter of fact, the median Grade IVB child was 99 per cent as accurate, and the Q_1 child in Grade IVB was about 93 per cent as accurate as they were likely to be a year and a half later in Grade VA.

The curves for speed are different from those for accuracy. (1) Whereas the greatest gain in accuracy occurred in Grade IVB (with the completion of the tables), the greatest gain in speed came a half year later, during Grade IVA. And this is as it should be, for educational theory has long held that accuracy should be sought before speed. (2) There was a distinct loss, rather than the expected gain, in rate of work in Grade VB. But this loss is not inexplicable. During the month and more which preceded the December tests these children had had little systematic work in multiplication, most of their time being spent on addition and subtraction with fractions. Under these conditions accuracy on the multiplication combinations might (as it did) persist at a high level, while quickness in thinking products might temporarily suffer.

INTERVIEW DATA

Results by Half-Grades

Grade IIIA.—Table 6 summarizes the interview data for Grade IIIA in connection with the nine facts which had been taught before the testing. This table is read: Of the 148 children interviewed, 54 obtained answers for 5 × 7 by recalling meaningfully established associations (H), 15 by making use of rote memory associations (M), 10 by guessing (G), 19 by solving from better-known combinations (S), 11 by counting (C), 9 by first reversing to the more familiar form 7 × 5 (R), 5 by reciting all or part of the table for 5 (T), and 25 made no attempt (N). The *columns* of data show the number of times different processes were used with each combination. The *rows* of data reveal the number of times a given kind of thought process was used with different combinations. Thus, there were 54 uses of H with 5 × 7, 40 uses with 8 × 3, 61 uses with 5 × 6, and so on, to yield a total of 514 times for the nine combinations.

In all, 1,332 responses are classified in Table 6 (9 combinations, 148 children[1]). Of this number, the largest single part are classified

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[1] Here and at other points the number of interview subjects exceeds the number of children whose group test data are reported. All (or practically all) children in each half-grade were interviewed in the local study, whereas sometimes test papers had to be discarded for reasons already mentioned.

TABLE 6

Local Study, Grade IIIA: Processes Used with Nine Selected Combinations* as Revealed in Interviews (N = 148)

Process†	7 5	3 8	6 5	6 4	3 7	9 2	4 7	5 3	3 6	*Total*
H	54	40	61	55	42	86	40	92	44	514
M	15	17	18	20	19	38	19	34	30	210
G	10	9	7	12	4	..	11	1	4	58
S	19	14	14	32	13	..	26	4	9	131
C	11	8	12	11	10	12	2	7	7	80
R	9	50	10	9	57	8	39	9	50	241
T	5	2	2	4	..	1	2	..	1	17
V	..	..	..	..	..	1	1	..	1	3
N	25	7	23	4	2	..	6	..	1	68
I	..	..	..	..	..	1	..	..	..	1
X	..	1	1	1	1	1	2	1	1	9

*In this place processes used on the taught combinations only are reported.

†In this table and in all similar tables the processes are referred to by code letter only. The code letters, as explained on page 52, above stand for:
H – Habituation, meaningful.
M – Memorization (little, or no meaning).
G – Guessing.
S – Solution (answer obtained through reference to better-known combination).
C – Counting.
R – Reversal (solution by reference to the paired combination).
T – Tables.
V – Visualization.
N – No attempt.
I – Indeterminate.
X – Miscellaneous (not susceptible to classification according to the categories above).

as H (514, or nearly 39 per cent). Not only is H the commonest response for the nine facts combined, but it is also the most popular for each separate fact. Next in popularity are M and R. M outranks R five times; and R outranks M four times, whenever the larger factor is the multiplier. Fourth in commonness is S (about 10 per cent of the total). Then come C, N, and G. None of the other processes total more than 1.3 per cent.

The figures in the table contain several facts of significance. First, a variety of mental processes were used with the nine taught combinations. Second, fewer than 40 per cent of the combinations were "known" as they are supposed to be learned under conditions of drill for mastery. Third, of the slightly more than 60 per cent dealt with in other ways than by H, one third were answered by undesirable procedures (M and G), and a few more than half by desirable but immature processes (C, T, S, and R). Fourth, the fact that 5 per cent were not attempted at all may be interpreted either as a healthy sign (children would not guess) or as an unfortunate condition (they had no way to find answers which they could not announce immediately).

Fifth, answers were seldom obtained by reciting the tables (T), the "table error" which has been mentioned in the literature as a "bad habit" commonly acquired by children in learning the multiplication combinations. This process was employed not at all with two facts, once with two others, and as many as five times (out of a possible 148 times) with only one fact, 5 × 7. Yet, the textbook studied by these children develops and teaches the combinations in tables.

Grade IVB.—Table 7 for Grade IVB is similar in construction to Table 6 for Grade IIIA, save that from this point on fifteen facts instead of nine were included in the interviews.

In Grade IVB, as in Grade IIIA, the columns of figures show that with each combination a variety of processes was used. Furthermore, the relative frequency of use of the commoner processes is about that of the earlier half-grade, with H first in popularity, M second, S third, and R fourth. The processes designated as V, I, and X occurred too infrequently to require comment.

TABLE 7

Local Study, Grade IVB: Processes Used with Selected Combinations as Revealed in Interviews (N = 136)

Process	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7 6	9 2	4 7	5 3	8 6	7 9	3 6	*Total*
H....	79	56	47	78	52	79	60	67	68	93	55	91	63	46	57	991
M....	20	18	15	16	17	15	16	24	13	18	16	18	17	27	19	269
G....	7	10	6	4	21	4	2	12	14	2	7	1	13	14	2	119
S.....	11	10	17	22	14	18	15	11	20	6	19	14	15	19	18	229
C.....	6	4	4	3	5	7	4	6	4	10	4	6	11	4	6	84
R....	5	24	41	5	9	5	37	3	4	4	26	2	3	12	30	210
T.....	5	4	5	2	8	6	2	1	4	3	6	2	2	..	3	53
V.....	1	1	..	1	..	1	..	1	..	..	..	1	2	1	1	10
N....	1	6	..	2	8	..	..	8	5	..	..	..	7	10	..	47
I.....	..	1	..	..	..	..	..	1	1	..	..	..	..	..	..	3
X....	1	2	1	3	2	1	..	2	3	..	3	1	3	3	..	25

H was used in nearly 49 per cent of the possible cases, as compared with about 39 per cent in Grade IIIA. This gain of ten percentage points may not seem large in view of the length of time these children had been studying the combinations. Progress toward general use of H was of course somewhat impeded by the introduction of sixteen new and more "difficult" combinations (six of which appeared in the interview). With the new combinations H constituted 43.1 per cent of the processes used; with the older combinations, 52.2 per cent. Not much gain therefore took place in use of H with the older combinations. Instead, for some reason there was a considerable increase in use of R with the four older combinations in which reversals could be of service, namely, 8 × 3, 7 × 3, 7 × 4, and 6 × 3. The chief source of improvement in Grade IVB as compared with Grade IIIA was a somewhat general desertion of the less desirable types of process.

Grade IVA.—The most noteworthy features of the interview data for Grade IVA, summarized in Table 8, are (1) the large increase in use of H (from 49 per cent in Grade IVB to 75 per cent in Grade IVA); (2) the virtual disappearance of V, I, N, and X; (3) the great reduction in use of the undesirable processes M, G, and N ("undesirable" at *any* stage in learning); and (4) the persistence of the immature processes S, C, R, and T, which should be undesirable by the end of Grade IVA. The processes grouped together in (3) above constituted 16 per cent, and those grouped in (4) about 9 per cent of the total for the half-grade.

TABLE 8

LOCAL STUDY, GRADE IVA: PROCESSES USED WITH SELECTED COMBINATIONS AS REVEALED IN INTERVIEWS (N = 136)

*Process**	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7* 6	9 2	4 7	5 3	8 6	7 9	3 6	*Total*
H....	97	92	100	106	90	104	107	96	98	113	108	111	98	91	109	1,520
M....	27	21	17	19	16	21	19	28	23	18	16	17	16	18	18	294
G....	2	5	1	..	8	1	..	1	..	..	2	..	4	8	..	32
S.....	7	9	8	7	9	3	2	7	9	1	6	1	7	9	4	89
C.....	..	1	1	1	1	1	2	..	1	1	..	1	1	1	1	13
R.....	3	4	6	2	7	5	3	2	2	3	4	3	4	4	3	55
T.....	..	2	3	1	2	..	2	2	2	..	..	3	3	3	1	24
V.....	..	..	..	..	..	1	1	..	..	..	..	..	1	..	..	3
N....	..	2	..	..	3	..	..	..	..	..	..	..	2	2	..	9

*"I" is omitted because there were no instances of this process, and "X" is omitted because there was but one instance, this in connection with 6 × 7.

Grade VB.—The figures for Grade VA (Table 9) reveal for the most part only minor changes from the situation in Grade IVA. There was no increase in use of H, but there was a considerable drop in use of M, which was however balanced by an unexpected rise in the use of R. S continued to be used to about the same extent as in Grade VA, and there was a very small and probably unreliable increase in the practice of reciting tables.

Grade VA.—The course of improvement, interrupted in Grade VB, was resumed in Grade VA (Table 10), where H alone accounted for 88 per cent of all processes used. Only one other process, M, appeared in more than 2.5 per cent of the occasions. Practically all of the undesirable and inefficient processes, M excluded, were about gone. And yet these Grade VA children had not yet achieved the

TABLE 9

Local Study, Grade VB: Processes Used with Selected Combinations as Revealed in Interviews (N = 138)

*Process**	7 5	6 9	3 8	6 5	8 7	6 4	3* 7	9 8	7 6	9 2	4* 7	5 3	8 6	7 9	3 6	*Total*
H....	113	96	89	117	107	115	100	113	110	120	93	119	110	101	102	1,605
M....	8	8	7	5	8	7	5	9	8	7	6	7	7	6	7	105
G.....	..	1	1	1	2	..	..	2	2	..	..	..	..	2	..	11
S.....	4	9	4	6	11	7	2	7	9	2	12	2	12	5	7	99
C.....	1	..	1	2	..	1	..	..	..	2	..	3	2	..	..	12
R....	4	18	31	4	5	5	28	3	5	4	21	5	2	20	20	175
T.....	6	3	5	2	4	2	1	3	3	3	4	1	4	3	1	45
N....	1	1	..	..	1	1	1	1	1	..	1	1	1	1	1	12
I.....	1	2	..	1	..	..	..	..	..	..	..	..	..	..	..	4

*No instances of "V"; 2 instances of "X," once each with 7 × 3 and with 7 × 4.

highest levels in abstract thinking and in understanding of the interview combinations (and supposedly of the combinations in general). It is to be noted that 12 per cent of the combinations were still dealt with by processes short of H.

TABLE 10

Local Study, Grade VA: Processes Used with Selected Combinations as Revealed in Interviews (N = 138)

*Process**	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7* 6	9 2	4 7	5 3	8 6	7* 9	3 6	*Total*
H....	122	114	117	127	116	124	117	116	120	128	122	127	122	120	124	1,816
M....	11	12	10	9	9	9	9	9	7	10	7	10	9	8	9	138
G....	1	1	..	..	4	..	..	..	1	..	1	..	1	..	..	9
S.....	1	2	2	1	7	3	3	6	5	..	..	..	4	6	..	40
R....	1	8	9	..	1	2	9	4	3	..	6	1	1	2	5	52
T.....	2	1	..	1	1	..	..	3	1	..	2	..	1	1	..	13

*No instances of "V," "N," "I," or "X"; 2 instances of "C," once each with 6 × 7 and 9 × 7.

Progress in General from Grade to Grade

It is rather difficult to secure from the detailed interview results for the successive half-grades a connected account or a satisfactory picture of the course of learning. For this reason Table 11 and Figure 2 have been prepared, the latter based upon the figures in the former. In both table and figure the separate combinations are disregarded, and the frequencies of each process are combined for all combinations and are then transmuted into per cents.

The interview data both supplement and correct the conclusions about learning which were obtained from the group test data. Where they supplement, they provide reasons for the facts as discovered. Where they correct, they make good the imperfections of the criteria of efficiency (rate and accuracy of performance). Let us begin with the "correction."

From Table 4 one gets the impression that learning of the combinations was about completed in Grade IVB and was certainly completed in Grade IVA. Here the criterion was accuracy. From Table 5 one gets the impression that, according to the criterion of rate, learning was about complete in Grade IVA (except for a temporary retrogression in Grade VB). But Table 11 and Figure 2 show unmistakably that learning was not ended at these earlier points—that, indeed, learning was not completed at the end of Grade VA, for possibly ten answers out of a hundred (twelve out of a hundred in the case of the *interview* combinations) were still obtained by processes short of the ultimately desired H. In a word, while in Grade VA answers were given promptly and with a high degree of accuracy, an appreciable number of children had yet to attain the final stages of abstract and mature thinking.

The interview data "supplement" the rate and accuracy data by explaining what lay back of the latter. This fact is best demonstrated by considering the results half-grade by half-grade.

TABLE 11

LOCAL STUDY: PER CENTS OF ANSWERS OBTAINED BY VARIOUS PROCEDURES

Type of Process	*Percentage Distribution, by Half-Grades*				
	*IIIA** (*N=148*)	*IVB* (*N=136*)	*IVA* (*N=136*)	*VB* (*N=138*)	*VA* (*N=138*)
Habituation, Meaningful...	38.6	48.6	74.5	77.5	87.7
Memory, Rote..	15.8	13.3	14.4	5.1	6.7
Guessing.......	4.4	5.8	1.6	0.5	0.4
Solution........	9.9	11.2	4.4	4.8	1.9
Counting.......	6.0	4.1	0.6	0.6	0.1
Reversal.......	18.1	10.3	2.7	8.5	2.5
Tables.........	1.3	2.6	1.2	2.2	0.6
Visualization....	0.2	0.5	0.1	0.0	0.0
No Attempt....	5.1	2.3	0.4	0.6	0.0
Indeterminate ..	0.1	0.1	0.0	0.2	0.0
Miscellaneous...	0.7	1.2	0.0	0.1	0.0
Total..........	100.2	100.0	99.9	100.1	99.9

*Processes classified relate only to the nine combinations (through 5's and reverses) supposedly mastered at the time of the interviews.

The results of the interviews in Grade IIIA agree with the group test results in showing that the experimental subjects were still some distance from mastery of the combinations they had studied. But they do much more than merely to agree with the test results: they *account* for them. The Grade IIIA children were slow and inaccurate on the group tests *because* they had meaningful habituations for considerably fewer than half the (interview) combinations and *because* they had to employ with combinations which they had not habituated, processes far short of H in maturity and efficiency. These processes, notably guessing, were susceptible to error, and they were employed clumsily and none too successfully, with the consequence of increased time and mistakes.

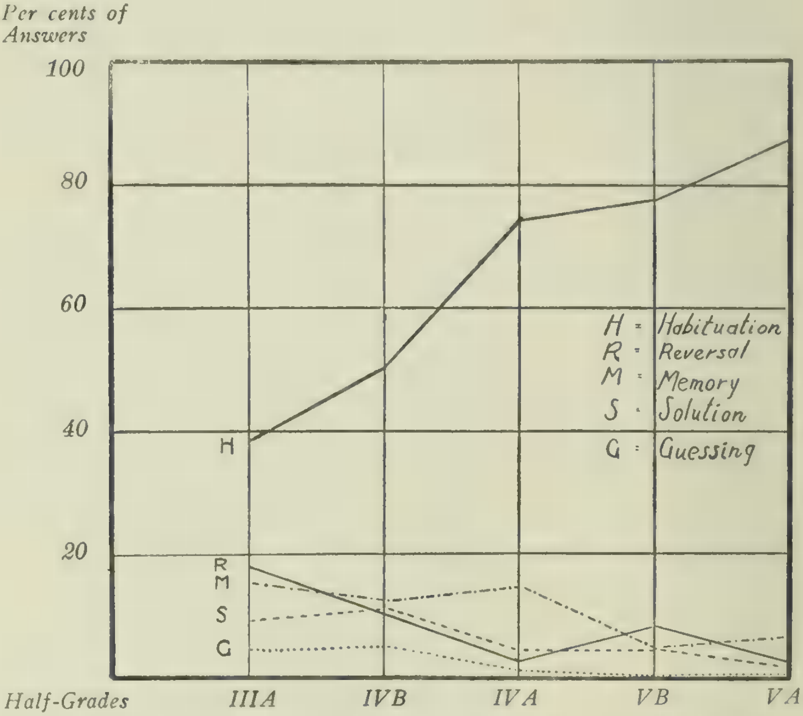

FIG. 2. Local study: changes in processes used from grade to grade, as revealed in interviews.

The rate curves for Grade IVB (Fig. 1) show moderate increases, and the accuracy curves for the same half-grades considerable increases. Improvement is to be explained, less by the abandonment of immature and inefficient processes than by greater control over the same processes which characterized the children in Grade IIIA. That is to say, the children in Grade IVB used the processes of Grade IIIA, but used them more effectively. Use of H gained only from 39 to 49 per cent. True, M decreased somewhat in popularity, but the relatively immature process of S became slightly commoner.

In Grade IVA the gains in accuracy brought the experimental subjects practically to the point of proficiency evidenced in Grade VA. The amount of this gain in accuracy over the Grade IVB showing was not great, but the amount of gain in speed of response was very marked. Again, the interview data provide the reasons. Three fourths of the answers were now obtained by H; one fourth more than in Grade IVB. Only one answer in ten was secured by immature (but desirable) processes, and only one in six (compared with one in five in Grade IVB) by undesirable processes.

Accuracy showed no improvement or loss in Grade VB, but there was a loss in rate. The changes in process were insignificant, save for a drop in M (undesirable process) and slight rises in R and S (both desirable, but both immature). The maintenance of the Grade IVB standing in accuracy is consistent with the lack of progress toward maturer thought processes, but the cause of the loss in rate had to be sought elsewhere. It was found in the absence of multiplication practice during the period preceding the group testing.

The Grade VA children had practically ridded themselves of immature processes (5 per cent) and of undesirable processes (7 per cent) on the interview combinations, and the effect of their relatively advanced thought processes is reflected in an average of 99 per cent accuracy and an average rate of two minutes, fifty seconds on the group test. The fact that inferior processes were still used to some extent indicates that there was yet opportunity for progress.

Progress toward Maturity of Process as Shown by Spaced Interviews

The reader may already have suspected more significance in the interview data than has yet been given them in this discussion. For one thing, the interview data show that the children in the local study did not learn the combinations as they were supposed to learn them. They were taught the combinations by drill, but they did not learn them by repetition—or at least by repeating what they were expected to repeat. Under conditions of drill, sight of "3 × 7" or "$\begin{array}{r}7\\ \times\ 3\\ \hline\end{array}$," for instance, is expected invariably and immediately to set off the response "twenty-one." Drill is supposed to cause the immediate establishment of such direct associations between factors and products. Yet, these children, taught by drill, did not immediately and invariably think the products and only the products. In place of using the expected associations, they tended to arrive at answers by a variety of processes. The expected process, H, or possibly M, was, it is true, common from Grade IIIA on, but it had not entirely supplanted other processes even at the end of Grade VA.

In a word, these children did not learn the facts all at once—at one jump, as it were. Instead, they seemed to learn by a series of jumps. This series carried them from undesirable, or inefficient, or immature processes through other processes intermediate in desirability, efficiency, and maturity, to the final stage of meaningful habituation. This is the picture of learning revealed by foregoing analyses of the interview data, and it is the picture revealed again when the processes employed by the same children at different points in learning are examined.[2]

For this more intimate study the records of seventeen Grade IV children will be examined, and their processes in the December interview will be compared with their processes in the May interview. (As a matter of fact, children could have been selected equally well from Grade III, had two interviews been arranged, or from Grade V. In the latter case, however, progress toward maturity of thought process would not have been so easily demonstrated.)

The seventeen selected subjects were less bright and less successful in arithmetic than the average for their grade. In the fall testing on the multiplication combinations their median accuracy score was seventeen points below the class median of 80, and their median time record exceeded the class median by a half minute. The median of this group was seven months above the grade median in CA, and nine and six months, respectively, below the class medians in MA and AA. Since these children had plenty of opportunity for improvement, they seemed especially useful for the present purpose.

Table 12 contains the interview records for both fall and spring.

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[2] It cannot of course be claimed that this coure of learning obtains regardless of differences in programs of instruction.

TABLE 12

PROCESSES USED BY SEVENTEEN GRADE IVB PUPILS ON THE INTERVIEW COMBINATIONS WHEN PRESENTED TWICE, IN FALL AND SPRING

Combination	*Subjects, with Processes Used in Fall and Spring*																
	HK	*MS*	*EM*	*RH*	*MM*	*JM*	*CD*	*BR*	*GJ*	*SS*	*AM*	*CG*	*MR*	*RF*	*DH*	*AH*	*BG*
7......	C*	M	G	T	M	T	S	G	M	C	V	M	H	G	S	T	M
5......	H	M	M	G	M	H	S	H	H	H	H	H	H	H	S	H	M
6......	C	M	M	X	M	N	S	N	M	C	N	M	S	C	S	T	R
9......	H	M	M	G	H	S	S	H	H	H	H	M	H	H	T	R	M
3......	C	M	G	X	M	S	S	R	M	S	R	R	C	C	C	T	G
8......	H	M	M	G	M	H	H	M	H	H	H	H	H	S	T	R	R
6......	C	M	N	X	M	T	S	G	M	C	V	S	H	C	S	T	T
5......	H	M	M	M	H	H	S	S	H	H	H	M	H	H	H	H	M
8......	C	M	G	X	M	N	T	N	G	C	N	G	S	C	S	G	M
7......	H	M	M	S	M	S	M	S	S	G	H	M	S	S	T	S	R
6......	C	M	G	X	M	S	S	G	M	C	T	S	S	S	S	T	N
4......	H	M	M	G	M	H	H	H	H	H	H	H	H	H	T	H	S
3......	C	M	M	H	M	S	S	C	M	S	T	S	H	C	C	R	R
7......	H	M	M	M	H	H	H	H	H	H	H	M	H	H	T	R	R
9......	C	M	M	X	M	N	M	G	G	C	N	M	H	G	G	M	R
8......	H	M	M	M	M	H	H	S	H	H	H	M	H	H	H	M	M
7......	C	M	G	X	M	S	S	N	M	S	N	M	S	G	C	T	G
6......	H	M	M	S	M	S	H	S	H	H	S	H	H	H	T	M	R
9......	C	M	M	M	M	S	C	C	M	H	M	H	H	H	H	M	C
2......	H	M	M	M	H	H	H	H	H	H	H	H	H	H	H	H	M
4......	C	M	G	X	M	T	S	S	M	S	S	C	S	C	S	T	N
7......	H	M	M	M	H	H	H	H	H	H	S	H	H	C	T	R	R
5......	H	M	G	H	M	S	S	C	H	H	V	S	H	H	C	T	M
3......	H	M	M	R	H	H	H	S	H	H	H	H	H	H	H	H	M
8......	C	M	G	X	M	N	S	N	R	C	N	S	S	C	G	T	M
6......	H	M	M	M	M	C	H	H	H	H	H	H	H	H	H	S	M
7......	C	M	G	M	M	T	S	N	S	C	N	S	S	C	S	T	R
9......	H	M	M	M	M	T	H	G	H	H	S	M	H	H	T	M	R
3......	C	M	M	M	M	S	S	C	M	S	T	S	H	H	C	T	G
6......	H	M	M	H	H	H	H	S	H	H	H	H	H	H	H	R	M

*The upper letter in each pair refers to the process used on the given combination in the fall; the lower letter, to the process used on the same combination in the spring.

In the first column are listed the fifteen interview combinations. At the head of each of the other columns are letters to identify the subjects. Code letters are used in the body of the table to indicate the processes with each combination, the upper letter in each pair referring to the fall interview and the lower, to the spring interview.

TABLE 13

Comparison of Processes Used, Fall and Spring, with Same Seventeen Combinations, Obtained by Interviews

Fall Processes, with Frequency	*Spring Processes, Same Combinations; Frequency*						
	H	*S*	*R*	*T*	*C*	*M*	*G*
H (18)	16	..	1	..	..	1	..
S (49)	34	7	..	5	..	3	..
R (10)	3	1	3	..	..	3	..
T (21)	10	1	4	1	..	4	1
C (43)	33	4	..	3	1	1	1
M (58)	22	..	1	..	..	35	..
G (25)	8	4	2	..	..	11	..
N (18)	7	7	1	..	2	1	..
V (3)	3	..	..	..	..	..	..
X (10)	1	2	1	..	..	4	2
Total (255)	137	26	13	9	3	63	4

The data in this table are then reclassified in Table 13 in order to permit readier comparison.

The first two columns of Table 13 show the immaturity and inefficiency of the thought processes employed in the fall. Only 18 times did this group of children use H to obtain answers, and these 18 times represent but 7 per cent of all processes. The most popular method was M. S and C were next most common, after which, at a considerable difference, came G, T, and R. Undesirable processes (M, G, and N) accounted for 40 per cent of the responses to the combinations, and immature processes (S, R, T, and C) for 48 per cent.

The processes are arranged in the table in a way which facilitates the study of progress toward maturity of method. At the top and left is H. S, R, and T, all of them desirable (in early stages of learning) from the standpoint of meaning, though less mature than H, come next from the top and from the left. They are followed by C, less desirable, less economical, and certainly less mature. Then come M, undesirable as well as immature, and G and N, with the same characteristics, plus the fact that they do not yield accurate answers. V, used only three times, and X (unclassified) are placed at the bottom, V being immature though possibly not undesirable.

In the spring interviews N, V, and X had disappeared, thus betokening progress. There was a very great increase in H, from 18 to 137 uses, with a falling off in the case of S, T, and C, the last-named especially. An exception to the general advance toward mature methods occurs in the case of M, where there was a small increase. The reason for this increase (if, indeed, it is an increase and is not the result of errors in classification) is not at all certain. Thirty-five of the combinations with which M was used in the fall were dealt with in the same way in the spring, but the other twenty-eight uses of M were produced by changes from all other processes except V. Especially common were the changes from G to M. It is debatable whether, from the standpoint of meaning, M is much preferable to G; but from the standpoint of the child who has no rational procedure but who *must* have *some* method of getting answers, M is certainly preferable to G.

One of the interesting facts to be observed in Table 13 is the nature of the changes which occurred during the half-year between the interviews. H, for example, was the most stable process; little change occurred. Of course, little change would have been expected, since H represents the last step in meaningful development. Changes would have had to be in a backward direction, toward less mature processes. Two such changes are to be noted in the table, one to M and the other to R, which however is but slightly lower in the maturity scale than is H.

In contrast to H, S was not stable; only 7 of the original 49 uses remained unchanged. What happened to the other forty-two uses of S? It is sometimes held that children ought not to be allowed to "solve" combinations, for fear that the methods of solution will be automatized in place of meaningful and direct associations (H). The data in Table 13 do not bear out this prediction and warning. Thirty-four uses of S changed in the half year to H; for 5, T was substituted, and T may or may not have been less mature than the form of S for which it was substituted. Only 3 uses of S changed to M. This is not a bad record for S at all. On the contrary, S seems to be a sound basis from which to move to more mature methods. As one means of noting the serviceability of S as a temporary step toward mastery, the records of CD and of DH in Table 12 are recommended for study.

R, considerably less common than was S in the fall, likewise has a fairly good record. Of the seven changes from R during the time between interviews, four were in the direction of greater maturity (H and S).

The use of tables, T, like the use of S, has been decried by certain writers in the field of arithmetic. These writers have been so fearful of the consequences of T that they have even recommended that the practice of teaching the multiplication tables be abandoned in favor of teaching the facts in random order. But, according to Table 13, T does not seem to be undesirable. After a half year's time only one of the 21 combinations answered by T in the fall continued to be so answered, while fifteen came to be treated by more mature processes. In this connection the record of Subject AH, Table 12, is instructive.

Counting in Grade IVB does not seem to have prevented growth toward mature processes—again failing to support the predictions of some experts in arithmetic who would deny children this method from the outset. Admittedly, in Grade IVB C is somewhat immature, no matter how desirable at an earlier stage. Within the half year 33 of the 43 C-responses changed to H-responses. For what happened to particular individuals who were especially given to C in the fall, the reader may consult the records of Subjects HK, SS, and RF in Table 12.

The data for M, which was relatively stable, have already been commented upon. In the earlier place it was suggested that, if *correct* answers are memorized, the immediate outcome for the puzzled child is not unhappy.

The worthlessness of G as a foundation for learning the combinations is well disclosed in Table 13. Of the 25 combinations whose answers were guessed in the fall, only 8 in the spring had been mastered in the sense of H, and only six more were dealt with by the better methods of S and R. Eleven of the combinations had been memorized, probably as the easiest way out.

To conclude, the foregoing analysis of seventeen pairs of interview records results in an account of learning which is considerably at variance with some current conceptions of learning. The nature of this conception has already been briefly noted in the opening paragraphs of this section, to which the reader is referred; and it will be elaborated in the last chapter of this monograph.

Variations in Progress

There is no little danger that the account of learning set forth in the two foregoing sections may seem to have been oversimplified. To some, this account may have implied that all members of any school grade are at the same stage in thought process, that they employ this favored process regardless of combination, and that they move forward all together and at the same time to the next higher stage in thought process. If any such impression was suggested, it was done so unintentionally (perhaps through the use of averages), and it is certainly in direct violation to the basic data themselves. In these data, the unique feature is not uniformity, but variation. Anything else than variation would be extraordinary, in view of all that is known about the facts of individual differences.

Nevertheless, in spite of these reassuring statements—or, as a means af validating them—it may be wise to illustrate the existence and the extent of variation in the local study. If this is done, it will be impossible to take a naïve view of the course of learning and to disregard factors which limit the applicability of the present account to individual classes, combinations, or children.

Variations within the same class.—There was a great deal of variation in each half-grade, in the extent to which a given process was used with different combinations. It is true that in Table 11 and in Figure 2 it was shown that H was responsible for 39 per cent of all answers in Grade IIIA; but 39 per cent is an average. Actually, H was employed to varying degrees with different combinations. That this variation was the general rule and that it held not merely for H but for all processes, and not merely for Grade IIIA but for all half-grades, is clearly indicated in Table 14. The reader can see for himself how frequencies of use for the commoner processes differed among the eight selected combinations. For example, in Grade IVB H occurred 58 times with 5 × 7 and only 35 times with 8 × 3; M occurred between 10 and 15 times, and so on.

Figure 3 shows even more strikingly the range in use of the commoner processes in each half-grade. In this figure, which is based upon Tables 6 to 10, the heavier parts of the horizontal lines indicate the ranges in per cents of use in each half-grade. For example, use of H ranged from 27 per cent to 58 per cent in Grade IIIA, from 34 to 69 per cent in Grade IVB, and so on. It is evident in Figure 3 and from Table 14 that extent of use of each process was quite variable. The statement that such-and-such a per cent of a given half-grade's answers were obtained by this or that process should not therefore mislead the reader into the impression of greater uniformity than actually obtained.

Variation for "easy" and for "difficult" combinations.—In this section the difficulty of combinations was determined, not from the group test data, but from the relative prevalence of H as a method

TABLE 14

VARIATIONS IN PER CENTS OF ANSWERS OBTAINED BY SAME PROCESSES WITH EIGHT COMBINATIONS, GRADES IIIA TO VA

Grades, with Principal Processes		*Particular Combinations (Each Second One in Test)*							
		7 5	3 8	8 7	3 7	7 6	4 7	8 6	3 6
Grade IIIA*	H	36	27	..	28	..	27	..	30
	M	10	12	..	13	..	13	..	20
	G	7	6	..	3	..	7	..	3
	S	13	9	..	9	..	18	..	6
	C	7	5	..	7	..	1	..	5
	R†	6	34	..	39	..	26	..	34
Grade IVB	H	58	35	38	44	50	40	46	42
	M	15	11	13	12	10	12	13	14
	G	5	4	15	1	10	5	10	1
	S	8	13	10	11	15	14	11	13
	C	4	3	4	3	3	3	8	4
	R†	4	30	7	27	3	19	2	22
Grade IVA	H	71	78	67	79	71	80	72	80
	M	20	14	12	14	17	12	12	13
	G	1	0	6	0	0	1	3	0
	S	5	5	7	1	7	4	5	3
	C	0	1	1	1	1	0	1	1
	R†	2	1	5	2	1	3	3	2
Grade VB	H	82	64	77	72	80	67	80	74
	M	6	5	6	4	6	4	6	5
	G	0	1	1	0	1	0	0	0
	S	3	3	8	1	7	9	9	5
	C	1	1	0	0	0	0	1	0
	R†	3	22	4	20	3	15	1	14
Grade VA	H	88	85	84	85	87	88	88	90
	M	[illegible]	7	7	2	5	5	7	7
	S	1	1	5	2	4	0	3	0
	R‡	1	7	1	6	2	4	1	4

*Only combinations taught in Grade IIIA are included.

†In Grade IIIA the average number of R's for combinations in which the smaller factor was the multiplier was 9; for combinations in which the multiplier was the larger factor, it was 49. The corresponding figures in Grade IVB were 4 and 28; for Grade IVA, 3 and 4; for Grade VB, 6 and 24; for Grade VA, 2 and 7.

‡"G" and "C" are eliminated in Grade VA because of infrequent occurrence.

for securing answers in Grade IIIA (later, Grade IVB). In Grade IIIA, H occurred most commonly with 3 × 5 (92 times) and with 2 × 9 (86 times); these two were selected as the "easiest" combinations. The "most difficult" combinations were 8 × 3 and 7 × 4, with which H occurred only 40 times each.[3] In Table 15 the figures

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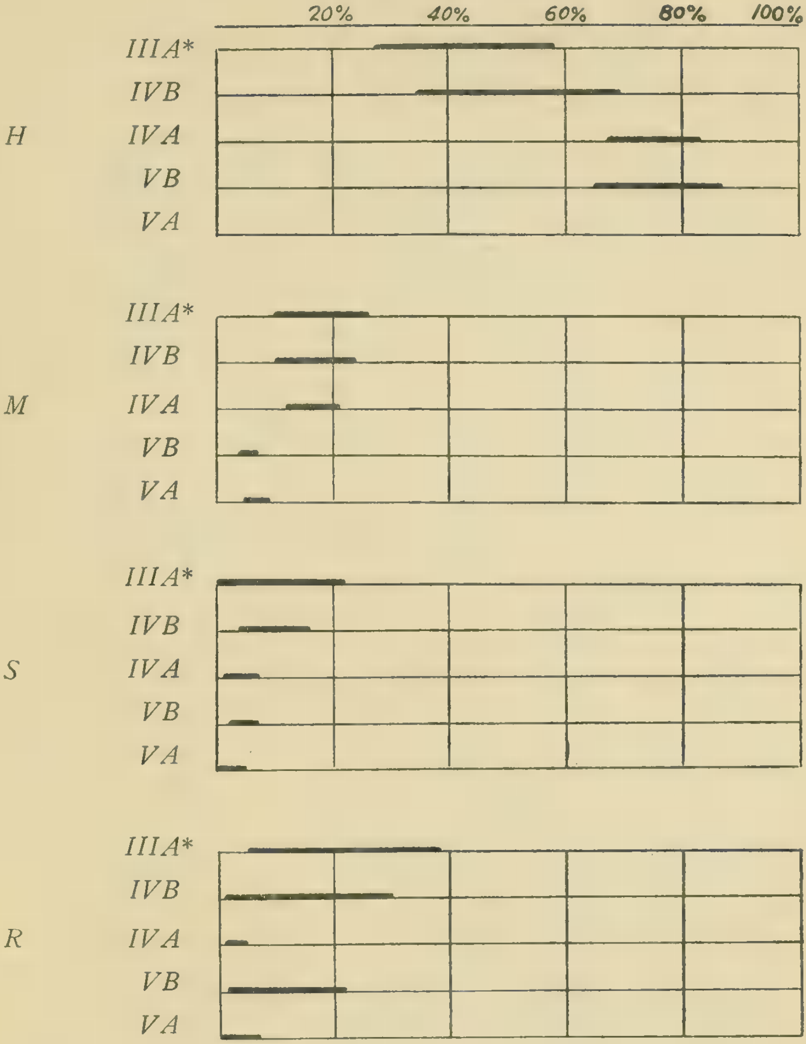
[3] According to the Norem-Knight rankings 2 × 9 is 47th in difficulty, and 3 × 5, 49th. The corresponding ranks for these two combinations according to Clapp are 13th and 33d. Norem and Knight rank 8 × 3 as 71st in difficulty and 7 × 4 as 76th. Clapp's rankings are 35th and 69th, respectively.



* Only the nine combinations taught in Grade IIIA are included for this grade. Calculations for other grades include all fifteen combinations in the interview test.

FIG. 3. Range in use made of the four principal processes within the same grades.

are expressed in terms of percentage, in order to permit comparisons of half-grade with half-grade. (The totals of the columns range from 98 per cent to 101 per cent because of the rounding of decimals.)

Even a casual glance at the figures in Table 15 reveals that the

TABLE 15

Comparison of Processes Used Throughout Five Half-Grades in Two of the "Easiest" and Two of the "Most Difficult" Combinations Taught in Grade IIIA

Process	*"Easy" Combinations*										*"Difficult" Combinations*									
	2 × 9					*3 × 5*					*8 × 3*					*7 × 4*				
	III A	*IV B*	*IV A*	*V B*	*V A*	*III A*	*IV B*	*IV A*	*V B*	*V A*	*III A*	*IV B*	*IV A*	*V B*	*V A*	*III A*	*IV B*	*IV A*	*V B*	*V A*
H	58	68	83	87	93	63	66	82	86	92	27	35	78	64	85	27	40	80	67	88
M	26	13	13	5	7	23	13	13	5	7	12	11	14	5	7	13	12	12	4	5
G	0	1	0	0	0	1	1	0	0	0	6	4	0	1	0	7	5	1	0	1
S	0	4	1	1	0	2	10	1	1	0	9	13	5	3	1	18	14	4	9	0
C	8	7	1	1	0	4	4	1	2	0	5	3	1	1	0	1	3	0	0	0
R	4	3	2	3	0	6	1	2	4	1	34	30	1	22	7	26	19	3	15	4
T	1	2	0	2	0	0	1	2	1	0	1	4	1	4	0	1	4	0	3	1
V	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
N	0	0	0	0	0	0	0	0	1	0	5	0	0	0	0	4	0	0	1	0
I	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X	1	0	0	0	0	1	1	0	0	0	1	1	0	0	0	1	2	0	1	0

easy combinations were dealt with somewhat differently from the difficult combinations. Besides the relative difference in prevalence of H (the basis upon which the combinations were selected), there are noteworthy differences in the commonness with which M, G, S, and R occurred. Evidently the degree to which a given process was employed was not fixed and alike for all combinations, but, instead, varied considerably in response to factors which have not been isolated in this study.

TABLE 16

Comparison of Processes, Grades IVB to VA, in Two "Easy" and Two "Very Difficult" Combinations, the Latter Taught in Grade IVB

Process	*"Easy" Combinations*								*"Difficult" Combinations*							
	5 × 7				*4 × 6*				*9 × 7*				*9 × 6*			
	IVB	*IVA*	*VB*	*VA*	*IVB*	*IVA*	*VB*	*VA*	*IVB*	*IVA*	*VB*	*VA*	*IVB*	*IVA*	*VB*	*VA*
H	58	71	82	88	58	76	83	90	34	67	73	87	41	68	70	82
M	15	20	6	8	11	15	5	7	20	13	4	6	13	15	6	9
G	5	1	0	1	3	1	0	0	10	6	1	0	7	4	1	1
S	8	5	3	1	13	2	5	2	14	7	4	4	7	7	7	1
C	4	0	1	0	5	1	1	0	3	1	0	1	3	1	0	0
R	4	2	3	1	4	4	4	1	9	3	14	1	18	3	13	6
T	4	0	4	1	4	0	1	0	0	2	2	1	3	1	2	1
V	1	0	0	0	1	1	0	0	1	0	0	0	1	0	0	0
N	1	0	1	0	0	0	1	0	7	1	1	0	4	1	1	0
I	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0
X	1	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0

Table 16 is similar to Table 15, the difference being that two "easy" and two "difficult" combinations taught in Grade IVB were selected for comparison, and Grade IIIA is necessarily omitted. No comment on Table 16 is called for: the same variability of process with combination noted in the case of the Grade IIIA combinations is to be observed for these other four combinations.

Variations within the same child.—After the evidence of variation which has been presented in the preceding two sections it will come as no surprise that individual children tend to be variable in the processes they use. The data already presented in Table 12 (p. 71) are enough to make this point. There it may be observed that of the seventeen selected subjects only two, MS and MM, employed the same process (in their case, M) with all fifteen combinations in the fall interview. One other subject, HK, used M with fourteen facts and H with one. The rest of the fourteen subjects used three or more processes, four using three, four using four, five using five, and one (Subject AM) using six. The median number of different processes was four.

Facts like the foregoing show why it is impossible to describe any child as being at a given point of maturity with respect to his thought processes for all combinations. There are, of course, exceptions to this rule—children who have arrived at the last stage of development will naturally use H with all combinations. Other children, like HK or MM, settle upon some process short of H, such as S or M, and remain at that point so long as the process serves their ends. But in general children may employ relatively advanced processes with certain combinations and relatively immature processes with others. Consider, for example, the case of GJ, whose processes ranged all the way from the most mature (H, with 3 × 5) through M (ten combinations) and R (6 × 8) and S (9 × 7) to the very immature and undesirable G (7 × 8, 8 × 9). Clearly Subject GJ, like most of the subjects in Table 12, was at the same time at quite unlike points in the scale of maturity with respect to different combinations. The maturity level of the process used with a combination seems to be suited to certain aspects of the relationships represented in that combination. What these "aspects" are is unknown. Indeed, it is highly probable that they vary from combination to combination with the same child, and from child to child with the same combination.

The Course of Learning with "Easy" and "Difficult" Combinations

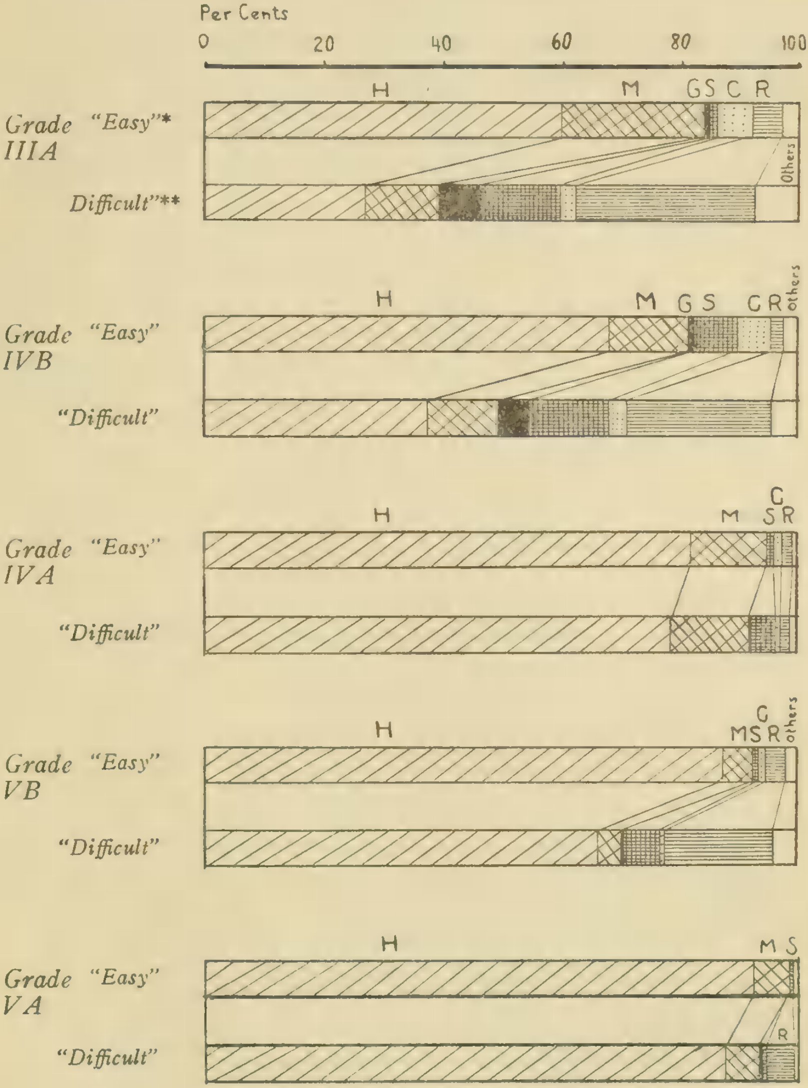
The fact of variation so amply demonstrated in the three sections immediately above in no way discredits the *general* account of learning represented in this chapter. That account is no less valid because each child pursues his own course in learning the combinations. One child's progress is, in its principal characteristics, like another's, but his rate of progress toward maturity may differ, as may also the details of his processes at any point in maturity. Recognition of variation then merely serves to forestall an oversimplified conception of what happens when a child learns the multiplication combinations—or learns any one combination, for that matter.

The same data which reveal variation may equally well be used to show the picture of learning as progress through a series of processes to eventual maturity and efficiency. Let the reader consider, for example, the data for the pairs of "easy" and "difficult" facts in Tables 15 and 16, data which have already been presented to show the fact of variation in process with combinations of different degrees of ease for learning.

Figure 4 is based upon Table 15. The first bar shows in percentage the relative use of the six major processes in learning the two "easy" combinations in Grade IIIA; the second, the corresponding data for the two "difficult" combinations. The next pair of bars is for Grade IVB, the next for Grade IVA, and so on.

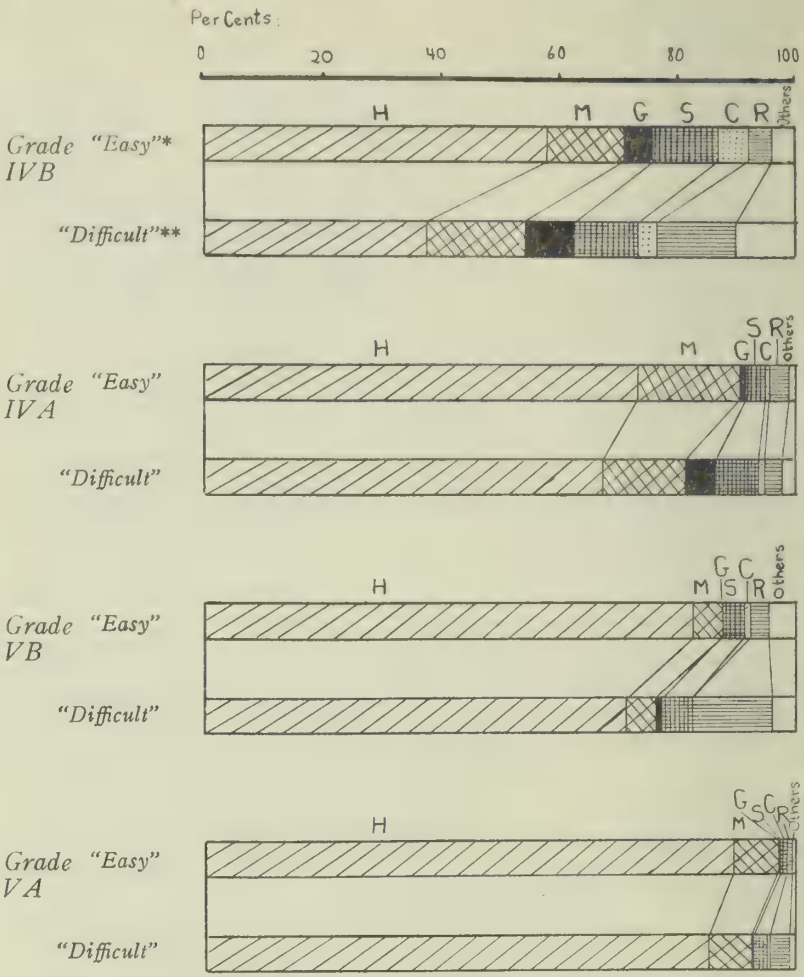
In Grade IIIA the space occupied by H in the bar for the easy combinations is almost twice that for H in the bar for the difficult combinations. In Grade IVB the difference in use of H for easy and for difficult combinations still persisted and was large in amount. In Grade IVA the margin of difference was greatly reduced, and by the end of Grade VA it was practically negligible. In other words, at this time H was used as commonly with difficult combinations as with easy combinations; it merely took the children longer to master the difficult combinations. The facts with regard to reduction in use of less mature and less efficient methods tell the same story: with the difficult combinations the rate of progress is merely slower; but progress is intrinsically of the same kind.

Figure 5 is based upon Table 16 and was constructed in the same way as was Figure 4. Different pairs of easy and difficult combinations are represented, but the picture of learning is identical with that shown in Figure 4. Difficult combinations were learned in the same way as were easy combinations; the transition to desirable, efficient, and mature thought processes was merely slower.



* "Easy" = average of responses for 2 x 9 and 3 x 5.
** "Difficult" = average of responses for 8 x 3 and 7 x 4.

FIG. 4. Percentage composition of answers by processes for "easy" and "difficult" combinations, by half-grades, IIIA to VA.



* "Easy" = average of responses for 5 x 7 and 4 x 6.
** "Difficult" = average of responses for 9 x 7 and 9 x 6.

FIG. 5. Percentage composition of answers by processes for "easy" and "difficult" combinations by half-grades, IVB to VA.

CHAPTER SUMMARY

The purpose of this study was to discover how a particular group of children, taught largely by methods of drill, learned the multiplication combinations. Data from individual tests were combined with data from group tests to achieve this purpose. First, the course of learning was plotted in terms of accuracy and rate of response. According to these measures of mastery, learning seemed to be complete by the end of Grade IVA—by the end of Grade IVB when the criterion was accuracy alone. Then the interview data were brought to bear upon the problem. They showed, first of all, that mastery was not attained in Grade IVA. Instead, children were found to be still using thought processes short of meaningful habituation and using them too often to warrant the conclusion that learning had been completed even at the end of Grade VA. The interview data were then found helpful in another way. They showed why changes in rate and in accuracy took place from half-grade to half-grade, or failed to take place in larger amounts.

The conception of learning which emerged from the above analyses is quite unlike the conception which is implicit in many educational programs based upon drill and which is explicit in certain psychological formulations. Children do not, as has been frequently suggested, either "know" or "not know" the combinations. Rather, they "know" them in different ways and to differing degrees, and these ways and degrees are identifiable in the processes which they use in thinking. The pages of this chapter are replete with examples of various kinds of processes.

It is dangerous at this point to generalize too freely from the interview data obtained in the local study alone; but at least an hypothesis with regard to learning may be suggested, especially since the next chapter will provide data by means of which to check this hypothesis. It is only when learning is viewed in terms of accuracy or in terms of promptness of answer that the illusion of quick, sudden learning arises. Actually children do not rapidly come to mature and efficient thought processes and hence to true mastery of the combinations. Instead, they move through a series of stages in thinking on the way to these final processes. In this connection pronounced individual differences were found in the local study, but they were differences in rate of progress. Some children seemed to move considerably faster than others and probably skipped a stage or telescoped two stages. Others moved slowly and probably stopped overlong at some intermediate stage. But such individual differences serve only to prevent an oversimplification of the picture of development; the picture itself is unchanged. It takes children a relatively long time to traverse the series of thought stages; but traverse them they must, if they are eventually to have useful and intelligent control over the combinations.

CHAPTER V

RESULTS OF THE EXTENDED STUDY

In the extended investigation, results of which are very briefly reported in this chapter, experimental subjects were drawn from an average of about twenty-five classes per half-grade, located in eighteen school systems and in eight states. The method of administering group tests and interviews was exactly like that in the local study except that teachers and administrative officers in the co-operating schools were responsible for these tasks.

The group test data will first be considered. Usable papers were obtained from slightly more than 600 pupils per half-grade on the average. Interview data were obtained from about 400 pupils per grade. In connection with these interviews it is probable that the attempt to distinguish carefully between the processes of H and M was less rigorous than in the local study.

GROUP TEST DATA

Accuracy Scores

The accuracy scores on the group test are assembled, half-grade by half-grade, in Table 17. At this point the distributions of scores in the two studies are compared only in a general way.

The accuracy scores in Grade IIIA were somewhat better in the extended study than in the local study. The medians compare 68 to 64, and the Q_1's, 61 to 55. In all probability the reason for this superiority lies in the fact that the pupils in the larger investigation had been taught more of the combinations before the test was given. More than a third of them had studied all eighty-one combinations, and about another third had studied the tables involving 8, 7, and 6—tables which had been withheld in the Burlington and Raleigh schools.

In Grade IVB the median in the extended study is 78, and Q_1 is 66. The corresponding measures in the local study, 80 and 75, are considerably better. But from Grade IVA onward, there is not much to choose between the two groups of subjects. In each half-grade in both studies the median is 81, though the per cents making this score average about eight points lower in the extended than in the local

TABLE 17

EXTENDED STUDY: ACCURACY SCORES IN GROUP TEST

Score	*Frequencies, by Half-Grades*				
	IIIA (*N=557*)	*IVB* (*N=613*)	*IVA* (*N=523*)	*VB* (*N=697*)	*VA* (*N=636*)
80	100	242	370	441	445
78	42	86	77	119	107
76	33	40	34	46	31
74	30	28	8	27	17
72	24	21	9	13	9
70	23	23	8	9	9
68	43	20	2	11	4
66	45	18	1	10	3
64	55	20	2	3	5
62	25	17	3	7	2
60	30	18	2	4	1
58	19	15	2	2	1
56	21	14	3	1	1
54	9	10	2	1	. .
52	13	8	. .	2	. .
50	8	6	. .	. .	. .
48	11	11	. .	. .	. .
47, and below	26*	16†	. .	1‡	1§
Actual Md	68	78	81	81	81
Actual Q_1	61	66	79	79	79

*46 (7); 44 (5); 42 (1); 40 (4); 38 (4); below 38 (5).

†46 (3); 44 (4); 42 (3); 40 (5); 38 (1).

‡46 (1).

§46 (1).

study. The Q_1's in these three half-grades were also one point lower in the extended study.

In view of the greater number of subjects represented and the greater variety of instructional programs followed in the schools of the extended study, it is probable that the medians and Q_1's in this larger study are more truly typical of the country as a whole than are the corresponding measures in the local study. Nevertheless, the discrepancies in the two studies are less striking than are the amounts of agreement.

Rate Scores

Table 18, for rate scores in the larger study, corresponds to Table 5 for the local study. According to the half-grade medians the subjects in the extended investigation worked slightly more rapidly in Grade IIIA (medians compare, 355 seconds to 380), at about the same rate in Grade IVB (285 seconds to 280 seconds), slightly

TABLE 18

EXTENDED STUDY: TIME RECORDS IN GROUP TEST

Time	*Frequency Distribution, by Half-Grades*				
	IIIA (*N*=557)	*IVB* (*N*=613)	*IVA* (*N*=523)	*VB* (*N*=697)	*VA* (*N*=636)
1:00	..	..	6	1	4
1:30	1	4	17	6	27
2:00	15	21	76	63	150
2:30	14	53	98	115	159
3:00	45	70	91	157	127
3:30	36	63	82	107	67
4:00	48	68	51	75	50
4:30	48	67	40	47	13
5:00	34	58	29	42	16
5:30	37	55	12	24	13
6:00	29	37	2	26	4
6:30	37	26	3	12	4
7:00	22	22	3	8	..
7:30	31	17	3	4	1
8:00	25	17	1	3	..
8:30	20	11	3	5	..
9:00	16	11	4	2	..
9:30	13	2	1	..	..
10:00	17	4	1	..	1
10:30	10	3	..	..	..
11:00, and over	59*	4†	..	..	..
Actual Md	5:55	4:45	3:20	3:20	2:40
Actual Q_1	8:25	5:55	4:10	4:25	3:45
Actual Q_3	4:20	3:35	2:40	2:50	2:25

*11:00 (14); 11:30 (8); 12:00 (14); 12:30 (5); 13:00 (8); 13:30 and over (10).

†11:00 (1); 11:30 (2); 12:30 (1).

slower in Grade IVA (200 seconds to 175 seconds), and again at about the same rate in Grade VB (200 seconds to 205 seconds) and in Grade VA (160 seconds to 170 seconds). The Q_1's in the two distributions are within ten seconds of each other, except in Grade IIIA, but there is somewhat less agreement in the case of Q_3's, the discrepancy being generally about twenty seconds and regularly in favor of the local study. Where the differences in measures in the two studies differ at all significantly, those for the larger study should probably be accepted as of more general validity.

Accuracy and Rate Considered Together

Figure 6 enables the reader to see where the gains in rate and accuracy were made in the span of the five half-grades. As before for the local study, the measures for Grade VA are taken as the standards, and the records in each half-grade are expressed as per cents of these standards.

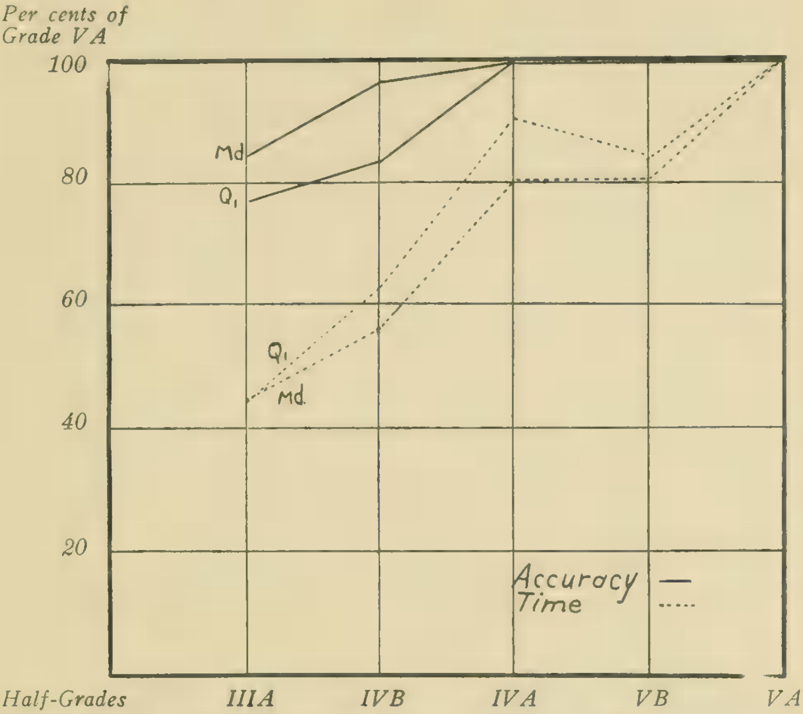


FIG. 6. Extended study: increase in accuracy and rate by half-grades expressed in terms of Grade VA medians and Q_1's.

As was noted in the corresponding curves for the local study, the greatest gain in accuracy took place in Grade IVB, and the accuracy standard of Grade VA was attained in Grade IVA. The pitch of the curve (median) is less steep between Grade IIIA and Grade IVB in the larger study. The probable reason has already been mentioned, namely, the exposure of the subjects in this study to a greater number of the multiplication facts in Grade IIIA.

The rate curves for the two studies are remarkably similar in slope. In both instances the greatest gains appear in Grade IVA, with little progress in Grade VB. The lag in rate as compared with accuracy is generally regarded as a desirable condition.

INTERVIEW DATA

The tables which report results of interviews in Grades IIIA to VA (Tables 19-23) will be found in the Appendix, pages 160-161. In point of construction these tables correspond in every detail with the tables for the local study, except for Grade IIIA. In the case of this grade in the local study processes were summarized only for the nine interview combinations which had been taught. In the extended study, on the other hand, the various classes had carried their study of the combinations to differing points—140 third-grade pupils having completed all tables, fifteen more all combinations in the tables through those for 8 and for 8's, sixty-four more through the tables for 7 and for 7's, and so on. Since each third-grade teacher reported the number of facts taught, it was possible to select from the fifteen interview combinations the ones which had been presented to each group and to tabulate the processes employed therewith. As a consequence, the total number of processes classified varies from 155 (for 8 × 9) to 474 (for seven combinations). These variations introduce no difficulty in interpretation. The relative frequency with which the different processes were used for a given combination can always be ascertained by reference to the total number of attempts with the combination, and the totals needed for these comparisons appear in the last row of Table 19. The per cents of use so obtained are then directly comparable, combination with combination.

The main facts with regard to processes used in the interviews closely parallel the corresponding facts obtained in the local study. These facts may be summarized in a series of short paragraphs. For support for each summary statement the reader is referred to the basic tables in the Appendix.

1. A great variety of processes were used with the combinations in each half-grade. Not even in Grade VA were all answers obtained by H; instead, immature, inefficient, and undesirable procedures were still in use in this grade. Actually the Grade VA children employed H to get only two thirds of their answers. One third of their answers were secured by processes short of H.

2. The extent to which given processes were employed varied largely from combination to combination. For purposes of illustration two processes may be selected, namely, H, the best, most mature, and the most efficient process on the one hand, and G, the poorest, least mature, and least efficient on the other hand. The following figures

are expressed in terms of the per cent of all processes represented by the two selected processes:

	Range in Use of H, in Per Cent	Range in Use of G, in Per Cent
Grade IIIA	38 (8 × 9) to 67 (2 × 9)	0.8 (2 × 9) to 21 (9 × 6)
Grade IVB	36 (7 × 8) to 73 (2 × 9) (9 × 7)	0.6 (2 × 9) to 21 (7 × 8)
Grade IVA	54 (9 × 6) to 79 (3 × 5)	0.9 (3 × 5) to 7 (7 × 8)
Grade VB	48 (9 × 7) to 74 (3 × 5)	0.7 (2 × 9) to 8 (7 × 8) (3 × 5))
Grade VA	55 (9 × 7) to 79 (3 × 5)	0.0 (2 × 9) to 5 (7 × 8) (3 × 5)

3. The occurrence of the four categories, V, N, I, and X (miscellaneous), was negligible. As for I and X, the frequency never was as large as 1 per cent for a combination and the inference is that interviewers were able to identify and classify practically all children's reports of their processes. The slight use of V (visualization), never as large as 1 per cent for a combination, is somewhat surprising in view of the opinion sometimes expressed that children rely upon visualization for their arithmetical thinking. As for N (no attempt), the statement previously made may be repeated: failure to try to answer may be either desirable as signifying a refusal to guess, or undesirable as indicating the possession of no means for obtaining answers. In either case, N ceased to be of much importance after Grade IVB.

4. T reached its greatest usefulness in Grade IVA, where it was responsible for 5 per cent of the answers. In no other half-grade did children recite tables to secure more than 2.2 per cent of answers. While these per cents may seem too large to warrant the statement that T is relatively insignificant among the habits used with multiplication facts, still they are so small as to suggest that the importance of this habit has been greatly exaggerated by those who oppose its use. The results in the extended study thus corroborate the findings in the local study, except that T played an even less prominent role in the latter investigation.

5. H was the most popular process from the start in Grade IIIA. At that time 45 per cent of the interview combinations were dealt with by this process, and this per cent became 50, 64, 59, and 66 in the succeeding half-grades. It may be assumed that for the eighty-one facts as a whole the per cents for H would be unquestionably higher, since the interview combinations are among the more difficult half of the facts.

6. Corresponding with an increase in use of H was a decrease in use of undesirable processes M and G. Guessing dropped from 8.2 per cent to 2.1 per cent, but was found still in occasional use in Grade VA. M likewise showed something of a drop, from 18.6 per cent in Grade IIIA to 16.4 per cent in Grade VA, but the abandonment of this process was less pronounced in the larger than in the local study.

7. The inefficient processes S, C, and R were also discarded along with the undesirable processes just discussed, but they did not wholly disappear: C yielded eleven answers per 1000 in Grade VA (down from forty-nine in Grade IIIA); S yielded thirty-nine answers (instead of eighty-five in Grade IIIA), and R yielded seventy-seven (compared with eighty-five in Grade IIIA). Extent of use of these processes is probably somewhat exaggerated in this investigation for the reason which has already been stated, namely, that the interview combinations as a whole are among the more difficult multiplication facts. In the case of R, six of the fifteen combinations selected for the interviews were peculiarly susceptible to the assistance which R affords. In every half-grade these combinations were dealt with by R to a disproportionate degree. For example, in Grade IIIA the ratio of use of R with these combinations (in which the first factor is the larger, as in 9 × 6) compared 2.5:1 with its use in connection with other combinations.

There are, then, differences in the extent to which the various processes were used in the local and in the extended studies, but these differences, with two exceptions, are minor and fall well within the expected limits of variation. The exceptions are the processes M and H. The following summary reveals the nature of the discrepancy:

	Combined M and H, in Per Cents		Ratio of H to M	
	Local Study	Extended Study	Local Study	Extended Study
Grade IIIA	53.4	63.1	2.5:1	2.4:1
Grade IVB	61.9	63.9	3.7:1	3.7:1
Grade IVA	88.9	76.2	5.2:1	5.2:1
Grade VB	82.6	78.8	14.8:1	2.9:1
Grade VA	94.4	82.2	13.1:1	4.0:1

In both studies the relative frequency of M and H combined increases from half-grade to half-grade (except for Grade VB in the local study), and at this point the results of the two studies are alike. But the total per cents differ markedly in three half-grades, M and H together having been used much less often in Grade IIIA and much more commonly in Grades IVA and VA in the local study. The greater weight should probably be attached to the figures for *totals* obtained in the extended study, in view of the larger number of pupils and the variety of instructional programs represented.

On the other hand, where there is a difference in the figures in the last two columns, those for the *local* study would seem to be more valid. The ratios of H to M are the same for the first three half-grades, but are quite unlike for the last two half-grades. It is hardly probable that the ratio of H to M should drop from 5.2:1 to 2.9:1 and to 4.0:1 in Grades VB and VA. Rather, the marked rise in this ratio found in the local study is much closer to reasonable expectation.

If the ratios for the local study are accepted for Grades VB and VA, the corresponding ratios for the extended study must be rejected. This statement amounts to saying that relatively too few uses of H and relatively too many uses of M were reported in the larger study. Yet, this hypothesis is not fantastic; nor is it a blind guess. On the contrary, it is quite likely. H-responses become steadily more difficult to differentiate from M-responses the higher the grade from which interview subjects are drawn. By the time children reach Grade V they have usually reduced their processes to automatic associations, and their answers based upon meaningful processes are distinguishable from responses of pure memorization only through careful and insightful questioning. The interviewers in the extended study were warned not to tabulate processes too quickly as H. This warning may well have overshot its mark, and interviewers may have failed to record as H processes that were actually of this type.

CONCLUSION

Chapter IV ended with a brief account of the way in which the children in the local study learned the multiplication combinations. In that place it was impossible to generalize and to say that the facts found are typical, that they are observable universally, for the basic data were obtained from children who had been subjected to but a single program of instruction. Now, the data in this chapter (Chapter V) corroborate the data from the local study in every important detail. In the larger as well as in the smaller study children are seen to have approached mastery of the combinations in a way which may be called indirect. True, when the course of learning is plotted in terms of rate and accuracy, learning seems to have been fairly direct; but not so when it is viewed in terms of thought processes used.

TABLE 24

EXTENDED STUDY: PER CENTS OF ANSWERS OBTAINED BY VARIOUS PROCESSES

Type of Process	*Percentage Distribution by Half-Grades*				
	*IIIA** (*N=155; 474*)	*IVB* (*N=483*)	*IVA* (*N=425*)	*VB* (*N=426*)	*VA* (*N=373*)
Habituation, Meaningful...	44.5	50.3	63.9	58.6	65.8
Memory, Rote..	18.6	13.6	12.3	20.2	16.4
Guessing.......	8.2	8.2	3.3	4.0	2.1
Solution........	8.5	4.4	2.5	2.7	3.9
Counting.......	4.9	4.1	1.3	1.5	1.1
Reversal.......	8.5	9.6	10.8	8.9	7.7
Tables.........	3.8	4.1	4.8	2.7	2.2
Visualization....	0.7	0.2	0.3	0.0	0.0
No Attempt....	2.0	4.3	0.7	0.4	0.3
Indeterminate ..	0.2	0.4	0.0	0.7	0.1
Miscellaneous...	0.2	0.8	0.0	0.1	0.2
Total.........	100.1	100.0	99.9	99.8	99.8

*See the explanation, page 88, for combinations included in Grade IIIA figures.

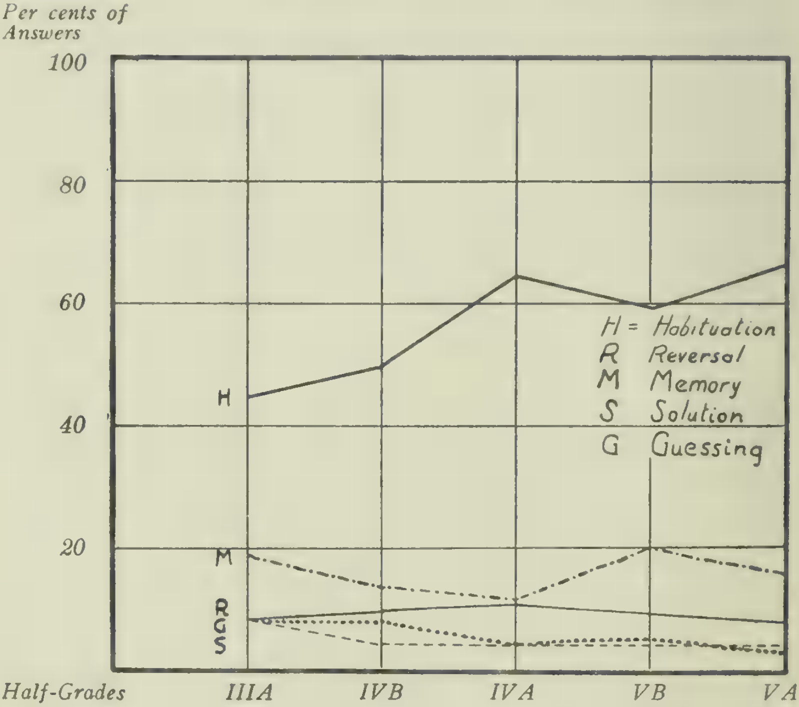


FIG. 7. Extended study: changes in processes used from grade to grade, as revealed in interviews.

Children at the early stages of learning, in Grade IIIA, reacted differently (by less desirable, mature, and efficient processes) from children closer to the point of intelligent mastery.

Table 24 and Figure 7, similar to Table 11 and Figure 2 for the local study, show the transition to mature, efficient, and desirable processes on the part of subjects in the extended study. Further comment on the course of learning is deferred to Part IV of the monograph, where the significance of the findings with regard to learning may be considered as they affect the teaching process.

CHAPTER VI

FACTORS WHICH MAY AFFECT LEARNING

Certain data collected by the interviews and the group tests in both studies are relevant to the question: What factors influence the learning of the combinations? "Learning" in this chapter, as elsewhere, is described in the familiar terms of the three criteria: rate of work, accuracy of answers, and method (or mental processes) employed in obtaining answers. The factors which are considered as possibly affecting learning in the three aspects just noted are: (1) chronological age (CA), (2) mental age (MA), (3) brightness (IQ), (4) general facility in arithmetic (AA) as obtained from scores on the arithmetic sections of the new Stanford Achievement Test, and (5) sex.

The plan of the chapter is as follows. Section A is devoted to a purely statistical study of the relationships between the learning criteria of rate and accuracy on the one hand and factors (1), (2), (3), and (4) on the other hand. The technique employed is that of correlation (Pearson Product-Moment Method). Section B explores the relationship between factor (5), sex, and all three learning criteria. Since sex is dichotomous, its relationship with the learning criteria is studied by means of measures of central tendency and dispersion (Md and Q).

A. CA, IQ, MA, AND AA AS FACTORS

The relationship between the four factors above and rate and accuracy of work is studied at three grade levels, IIIA, IVB, and VB, where complete records were available for at least 107 subjects each per half-grade.[1]

Time as the Criterion of Learning

Table 25 contains the coefficients of correlation between rate as the learning criterion and factors (1) to (4). None of the coefficients are large.

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[1] Intelligence tests were not given in Grades IVA and VA, though of course the Grade IVB and VB IQ's could have been used and new MA's found by extrapolation.

TABLE 25

Coefficients of Correlation Between Time and Various Factors Which Influence Learning

Items Correlated	*Grade IIIA*	*Grade IVB*	*Grade VB*
Time and CA	−.11	−.09	−.07
Time and IQ	.23	.18	.05
Time and MA	.22	.24	.08
Time and AA	−.28	.14	−.07

The negative coefficients for CA suggest a slight tendency for the older children in all three half-grades to take more time on the group test than did the younger children. The brighter children (IQ) took more time than the duller in Grade IIIA, but less time in Grade IVB. In Grade VB the relation for IQ is practically zero, probably because the greater familiarity of all subjects with the combinations had by this time erased any effects of differences in IQ. So much for the coefficients as such. The relationships between rate of work and the four factors considered are chiefly of academic interest. That is to say, the relationships are hardly of much significance from the standpoint of learning and teaching, because time to complete the test is a highly ambiguous criterion of learning.

Greater familiarity with the combinations may also have been responsible for the smaller relationship between MA and time (never a strong relationship) after Grade IVB. Proficiency in arithmetic (AA) seems to have had varying relationship with rate of work. The "best" time records could have been made by children who attempted to write answers for none of the combinations. "Good" time records could have been turned in by children who attempted only a few combinations, or who wrote as answers the first numbers that came into their heads. Conversely, "poor" time records could have been made by subjects who seriously tried to write correct answers for all combinations, some of whom might have had very high accuracy scores.

The correlation coefficients for rate of work and for accuracy scores were: −.04 in Grade IIIA, .57 in Grade IVB, and .33 in Grade VB. The zero relationship in Grade IIIA justifies the criticisms just raised with respect to rate as a measure of learning in that half-grade. The higher and positive correlation in Grade IVB is understandable, in view of the fact that this was really the "learning" half-grade; all children wrote as many answers as they could and those who "knew" the combinations best probably worked more rapidly. The lower relationship in Grade VB is again probably the consequence of the increased familiarity of all subjects with all combinations.

Accuracy as the Criterion of Learning

Accuracy of answer is a sounder criterion of learning than is rate of work, if for no other reason than that only correct answers could contribute to the scores used as measures of learning. At the same time, accuracy is not acceptable as the sole criterion of learning (though it is usually so accepted in the classroom), for it omits all consideration of the way in which answers are obtained.

Table 26 summarizes the coefficients of correlation between number of correct answers on the group test and four factors which may possibly affect learning. Unlike the coefficients in Table 25, the measures in Table 26 are mostly positive, and a larger per cent of them signify that relationships do exist between the learning criterion and the factors chosen for study.

TABLE 26

COEFFICENTS OF CORRELATION BETWEEN ACCURACY AND VARIOUS FACTORS WHICH MAY AFFECT LEARNING

Items Correlated	*Grade IIIA*	*Grade IVB*	*Grade VB*
Accuracy and CA	−.08	−.19	−.07
Accuracy and IQ	.45	.30	.14
Accuracy and MA	.51	.53	.08
Accuracy and AA	.59	.32	.03

All coefficients between CA and rate are slightly negative, signifying that in general the older children worked less accurately than did the younger children. Within the limits of each half-grade this negative relationship could have been predicted, for the older children are also the less capable children.

IQ seems to be moderately related to number of correct answers in Grade IIIA, considerably less so in Grade IVB, and much less so in Grade VB, where the relationship all but disappears. These figures would appear to mean that brightness is definitely related to the ability to give correct answers in Grade IIIA—it is, in other words, a real factor making for ease of learning at that time—but that its influence wanes steadily in face of increased experience and drill, until in Grade VB its importance in learning is negligible.

On the other hand, MA persisted as a factor in learning through Grade IVB, that is to say, until after all combinations had been presented for learning. The correlation coefficients for MA are especially interesting in view of the current proposal to determine the grade for teaching the multiplication combinations in terms of this factor. The significance of these coefficients in this connection will be considered in Chapter IX.

The highest correlation coefficient in Table 26 is .59, for accuracy and AA in Grade IIIA. It is not unreasonable to assume that general proficiency in arithmetic, as measured by AA, should have provided a good basis for learning the first multiplication combinations in that half-grade. The sharp drop in the size of the coefficients for Grades IVB and VB (in the latter grade the coefficient is zero) is not so easily explained, especially since both in the multiplication test and in the standard arithmetic test accuracy of work is a common factor. Possibly the drop in the coefficient means that increases in accuracy on the multiplication combinations from grade to grade become relatively less important in determining scores made on the test of general arithmetical achievement.

B. SEX AS A FACTOR

Sex as a factor in learning no longer holds so much interest for psychologists as it did three or more decades ago. Probably the reason for their declining interest lies in the consistent findings of research that in purely intellectual functions differences between the sexes are not large, that one sex achieves about as well as the other, and that, in terms of averages, from 40 per cent to 60 per cent of one sex usually equals or exceeds the average of the other sex.

In the case of arithmetic, the results of research have not always been in complete agreement. Two sample studies relating to the number abilities of children on entering school may be cited. Buckingham and MacLatchy[2] report a small but consistent superiority for girls. The tasks included in their individual tests comprised rote counting by 1's and by 10's, enumeration of objects to 20, reproduction of groups of objects numbering 5, 6, 7, 8, and 10, and various ways of dealing with selected addition and subtraction combinations. On the other hand, Woody (29), whose inventory contained many more number tasks than did Buckingham and MacLatchy's inventory, found a general superiority for boys. On the test as a whole the boys surpassed the girls in four of the five grade groups (kindergarten to

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[2] B. R. Buckingham and Josephine H. MacLatchy, "Number Abilities of Children When They Enter Grade One," *Report of the Society's Committee on Arithmetic* (Twenty-ninth Yearbook, National Society for the Study of Education; Bloomington, Ill.: Public School Publishing Co., 1930), pp. 473-524.

Grade IIA), although none of the differences were statistically reliable. The greater ability of boys was especially marked in the social applications of number, such as money, telling time, and the like.

So far as the multiplication combinations are concerned, Holloway (**10**: 98) reports that girls clearly excelled boys in two of his studies in their knowledge of the multiplication facts. Other investigators have not always agreed with this finding.

In the intermediate and higher grades the advantage has regularly been found to lie with boys both in abstract computation and, even more so, in problem solving. Furthermore, the amount of this advantage appears to increase from grade to grade.

No research, however, has been concerned with the reason why one sex surpasses or may be expected to surpass the other. Hypotheses have been advanced, but these have tended to be incompatible. The one hypothesis, the traditional hypothesis, assigns differences to biological and heritable causes—boys "naturally" have more "mathematical aptitude" or they "just naturally" are able to "think" better than girls. The second hypothesis attributes the cause of sex differences to unlike social or cultural factors. Advocates of the second view point to the excessive restrictions to which girls are subjected. As a result, girls early become submissive and docile, inclined not to challenge but to accept the statements of older persons, such as teachers, without examining into their rational bases. If the second explanation is correct, girls should be expected, but not for biological reasons, to approach intellectual learning tasks differently from boys, for example, to memorize in occasions in which boys "think." Differential effects in arithmetical learning would therefore be found in the way in which they learn,[3] whether or not in the breadth or depth of their learning.

Certain data collected in the present studies pertain to sex differences with regard both to method or way of learning and to extent of achievement. The data which relate to comparative achievement will first be considered.

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[3] Freeman briefly summarizes the findings of research on sex differences as follows:

"Practically all investigators have found a small but consistent superiority of females over males in linguistic ability. In number concepts and arithmetical ability, including computation and reasoning, although the results are not so unequivocal, the weight of evidence is on the side of a small difference in favor of the male group. In tests of memory, girls as a group appear to be somewhat superior to boys" (F. S. Freeman, p. 423, in Charles E. Skinner, ed., *Educational Psychology*, New York: Prentice-Hall, Inc., 1940).

Samples of subjects were selected from the populations of both the local and the extended investigations. In the case of the local study fifty consecutive records were drawn for both boys and girls separately in each half-grade. In the larger investigations every second record for boys and for girls was chosen until samples of one hundred records for each sex in each half-grade had been obtained. The procedure employed to secure the samples very probably precluded the operation of selective factors which could have made the samples unrepresentative.[4]

Tables 28 and 29 contain measures of comparative accuracy and of comparative rate of work, respectively, for the sex samples drawn from the two studies. In accuracy, the Grade IIIA median of the girls is six points higher than that of the boys in the local study, and three points higher in the extended study. Where there are differences, Q_1's and Q_3's also favor the girls. In Grade IVB the girls' advantage

TABLE 28

Measures of Accuracy on Combinations Test Made by Samples of Boys and Girls, Local and Extended Studies

Measure	*Grade IIIA*		*Grade IVB*		*Grade IVA*		*Grade VB*		*Grade VA*	
	Boys	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*
	(Local study, N=50 per sex)									
Q_1..........	54	60	72	78	80	80	80	80	80	80
Md.........	62	68	79	80	81	81	81	81	81	81
Q_3..........	72	76	80	81	81	81	81	81	81	81
Q...........	9.0	8.0	4.0	1.5	0.5	0.5	0.5	0.5	0.5	0.5
	(Extended study, N=100 per sex)									
Q_1..........	62	62	65	69	78	80	79	79	79	80
Md.........	67	70	78	79	80	81	81	81	81	81
Q_3..........	77	79	81	81	81	81	81	81	81	81
Q...........	7.5	8.5	8.0	6.0	1.5	0.5	1.0	1.0	1.0	0.5

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[4] A statistical study of the sex samples in the local study showed the following facts with regard to medians: in CA, girls and boys identical in Grade IVB and VB, with a difference of one month in favor of the boys in Grade IIIA; in MA, sexes identical in Grade IVB and within two months of each other in the other two half-grades; in IQ, girls superior by from one to four points in each half-grade; in AA, girls superior by six months in Grade IIIA but inferior by smaller margins in Grades IVB and VB. The only important difference then was in the case of AA in Grade IIIA; the differences in IQ were unreliable. There was no consistent tendency for girls or boys to be more or less homogeneous. (See Table 27 in the Appendix.)

In other words, the sex samples in the local study seem to have been very like each other. In view of the much larger samples in the extended study, it is reasonable to believe that the two corresponding sex samples were even more alike in the four factors mentioned above.

TABLE 29

MEASURES OF RATE ON COMBINATIONS TEST MADE BY SAMPLES OF BOYS AND GIRLS, LOCAL AND EXTENDED STUDIES

Measure	*Grade IIIA*		*Grade IVB*		*Grade IVA*		*Grade VB*		*Grade VA*	
	Boys	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*
	(Local study, N = 50 per sex)									
Q_1	5:25	5:00	3:40	3:25	2:35	2:25	2:50	2:45	2:30	2:25
Md	6:30	6:50	4:45	4:35	3:05	2:45	3:30	3:20	2:50	2:50
Q_3	7:55	8:55	5:45	5:20	3:55	3:25	4:00	3:45	3:30	3:25
Q	1:15	1:58	1:03	0:58	0:40	0:30	0:55	0:30	0:30	0:30
	(Extended study, N = 100 per sex)									
Q_1	4:20	4:15	3:40	3:35	2:35	2:35	3:00	2:50	2:25	2:20
Md	6:10	5:25	4:50	4:30	3:10	3:25	3:20	3:20	2:45	2:40
Q_3	8:45	8:15	5:55	5:50	4:10	3:55	4:20	4:25	3:50	3:35
Q	2:23	2:00	1:08	1:08	0:48	0:40	0:40	0:48	0:43	0:38

in medians is reduced to a single point (and hence is not reliable) in both studies, and from this point on the average achievement of the two sexes is identical.

In rate of response, the girls' medians surpass the boys' in three of the five half-grades in both studies, and equal or surpass them in four of the five half-grades. About the same amount of advantage holds for the girls when Q_1's and Q_3's are compared.

A fair summary of the situation is that at the outset girls surpass boys in "learning" the combinations, whether the criterion is accuracy or rate, that in accuracy they lose their advantage very soon, by the end of Grade IVA certainly, but that in rate of work they tend to maintain their advantage for at least a half-grade longer.

The data on processes used may now be considered. Table 30 summarizes the frequencies with which the different processes were used by the 150 boys and the 150 girls in each half-grade sample.

In every half-grade the boys surpassed girls in use of H, the disparity ranging from a difference of ten uses (Grade VB) to 144 uses (Grade IVA). On the other hand, girls used M more than did boys in every half-grade, by margins ranging between fifty-eight (Grade IVB) and eighty-three (Grade IVA) and eighty-two (Grade VA). They likewise employed R more commonly in four half-grades (Grade VB is the exception)—and it is to be remembered that by using R they may have been able to convert the reverse combinations to direct combinations with which they could have used M. In the two lowest half-grades where G occurred often enough to be called

TABLE 30

PROCESSES USED, WITH FREQUENCY, BY SAMPLES OF 150 BOYS AND OF 150 GIRLS DRAWN FROM BOTH LOCAL AND EXTENDED STUDIES

Process	*Frequencies by Half-Grades, Boys and Girls*									
	IIIA		*IVB*		*IVA*		*VB*		*VA*	
	Boys	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*
H......	849	821	1,263	1,225	1,654	1,510	1,531	1,521	1,692	1,603
M......	332	395	247	305	252	335	282	347	277	359
G......	195	213	120	125	35	28	51	46	24	41
S.......	226	174	169	114	76	56	91	73	68	66
C......	133	91	115	62	24	15	34	40	6	12
R......	157	213	202	247	187	226	189	164	133	139
T......	54	64	101	77	117	63	58	39	42	26
V......	2	2	11	1	1	4	...	...	...	2
N......	285	278	89	71	4	11	2	2	1	2
I.......	5	6	5	10	...	...	9	18	6	...
X......	8	22	12	10	...	...	8	16	6	...

common, girls used this process more often than did boys. In the same half-grades they employed S and C less frequently.

None of these differences when taken alone means very much; but when taken together these differences do seem to indicate that the sexes were not alike in their attack upon the combinations. In general, girls in the lowest half-grades relied more often than boys upon undesirable processes (M and G) as well as upon immature processes (R). Moreover, they clung longer to R and M and were slower to adopt H. Admittedly these sex differences are not large, and they certainly imply no qualitatively different approaches in learning. Nevertheless, the differences are consistent, and they warrant the tentative hypothesis that girls tend less commonly than do boys to approach the combinations on a basis of understanding.

If further research should establish this hypothesis as valid,[5] it is not unlikely that at the same time an explanation may be found for the failure of girls to maintain throughout the intermediate grades the initial superiority in arithmetic which they are reported as enjoying in the lower grades. The processes of M and R, for example, may give a temporary advantage in writing or saying products quickly and accurately, as in Grades IIIA and IVB, but they provide an insecure foundation for retention and for successful use of the combinations in the complex multiplication computation of higher grades. In the end the boys, who suffer in rate and accuracy comparisons in the lower half-grades, may actually be learning in the sounder manner.

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[5] The senior author is at the present time trying to get research organized for an attack upon this problem.

The present research justifies no more than the hypothesis which has just been advanced. It does not, for instance, afford any evidence on a second hypothesis which was mentioned earlier. There still remains the question why girls should use lower-order processes than do boys in learning the combinations (if, indeed, they do). The explanation may be biological: girls inherit less "mathematical aptitude" or inherit less of whatever eventuates in "mathematical aptitude." Or, the explanation may be cultural: in the way in which girls learn the combinations they merely reflect the effects of differences in social pressures and encouragements which are general in their application. The writers incline to the latter explanation.

Part III

Readiness to Learn the Multiplication Combinations

CHAPTER VII

THE PROBLEM OF READINESS

MEANING OF "READINESS"

Properly conceived, "readiness" to learn anything involves the joint action of a number of factors.[1] It obviously involves the intellectual ability to learn whatever is to be learned; but it involves much more than this. It involves the factor of motivation, the personal desire and intent to learn on the part of the learner. It may easily involve emotional factors, for example, the joy of successful achievement and, before this, intense eagerness to learn. It involves, or may involve, social factors, for the child's cultural milieu may present him with the need, even the necessity, of learning. It may involve obscure physiological factors, such as those represented in the requisite degree of development of the parts (receptors and effectors) essential to learning.

In educational literature, the term "readiness" does not commonly include all these factors. Instead, "readiness" means purely the intellectual ability to learn, and emphasis is put upon perceptual and conceptual processes, while the motivational, social, and physiological[2] factors are unrecognized or disregarded. In a way this narrow usage is unfortunate, for it encourages the practice of neglecting other factors which make for effective learning. Hence, the learner often is required to face tasks too soon and without adequate preparation and motivation.

On the other hand, there is some value in restricting the term "readiness" to the single (but by no means simple) factor of ability to learn. Such restriction has the advantage of centering attention upon one essential condition of learning and so of getting something done about it. This is not to say that other factors are less crucial to economical learning; rather, it implies that if "readiness" is so construed we should have another more comprehensive term to cover "readiness" in the narrow sense as well as all other factors. It is in this narrow sense that the term will be used (hereafter without quotes) in this monograph.

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[1] Clifford Woody, "A General Educator Looks at Arithmetic Readiness," *Mathematics Teacher*, XXX (1937), 314-321.

[2] The term "physiological factors" of course includes neural factors. In his subsidiary law of learning, the Principle of Readiness, Thorndike limited the word "readiness" exclusively to the condition of the neurones which were concerned in a given reaction. Whether this condition results from heredity, from previous learning, or from both depended, for him, upon the particular reaction and upon the particular neurones under consideration. In some of the more recent educational uses of "readiness" (for example, in many places in the reports of the Committee of Seven, to be considered shortly) there is the clear implication that "readiness," as for the multiplication and the addition combinations, for "long" division and the like, is dependent solely upon the maturation of highly specialized neural mechanisms. In this development experience seems to be regarded as inoperative.

NEWNESS OF THE PROBLEM

It was common practice until a decade or two ago to present all 100 multiplication combinations in Grade III. This practice was based upon tradition rather than upon research and implicitly involves the assumption that by the middle of Grade III children are ready to start the combinations and that by the end of the term they can complete the learning. More recently, the tendency has been to postpone a good many combinations to Grade IV, and in Grade III are taught only the tables through those for 4 or for 5 or all the combinations with products to 20, 25, or some such number.

One must concede that the older practice of teaching all the combinations in Grade III seldom produced the expected results. Scores on informal tests and on standard tests alike revealed lack of mastery at the end of Grade III. But such test scores could hardly reveal the *reason* for this lack of mastery. The commonest inference from the test results seems to have been that children had not learned the combinations because at that time (Grade III) they were not ready for them. This is but to say that they had not learned because they were then intellectually incapable of learning. Hence, as noted above, part of the combinations, if not all of them, were deferred to Grade IV or even later.

Undoubtedly, impetus was given this solution of postponement by the growing acceptance of the educational philosophy which is epitomized in the phrase "the child-centered school." The rediscovery of the child in the enterprise of education aroused new concern for him and for his sound, wholesome development. The traditional curriculum was viewed as an adult program which had been imposed upon the child without adequate consideration for his powers and his welfare. Under current conditions of instruction the average child found it difficult, if not impossible, to learn all the multiplication combinations in Grade III; hence, the way out was to defer at least some of them until a later grade.

But there is an alternative explanation for the poverty of learning when all the combinations were taught in Grade III. Perhaps children did not learn the combinations taught them because the combinations were not well taught. Children might possibly have been successful if their learning activities had been appropriate, appropriate alike to their powers and to the task at hand. This hypothesis, surely as tenable as that of sheer incapacity to learn, was not investigated; indeed, it was seldom seriously considered. The temper of the times was wrong: failure to learn was equivalent to inability to learn, and the difficult learning task was deferred. Educational theorists and practitioners were borne along in a wave of sentimentality with respect to childhood.

PURPOSE OF PART III

At the time this monograph is being written educational practices are being critically examined. We have not lost our interest in the child. The idea of the "child-centered school" has made its contribution, and it is an important and lasting one. We shall not in any likelihood revert to a curriculum or to a methodology which minimizes the place of the child in the educational picture. But many educators are questioning the wisdom of practices adopted in the throes of emotion—at the same time they are refusing to return to the "old" education. We have, for example, adopted the "stepped-up curriculum," in accordance with which learning tasks have been rather generally moved higher in the grades. Why did we adopt this program? Were the reasons sound and adequate? Or, were we misled by ambiguous terms ("readiness," "maturation") and a misconception of the nature of development? Does "maturation" apart from experience produce the ability to learn? If children, taught as we teach them, do not learn, must we wait until they are older to teach them? Or, is there something we can do about readiness? Can we influence readiness? By manipulating experiences can we actually get children ready for learning? If so, may not our methods of teaching be more significant than we have thought; and cannot we, by discovering better methods, do things we have thought impossible?

It is the purpose of Part III to consider such questions as these as they relate to the multiplication combinations. In Chapter VIII will be canvassed the evidence on readiness which has been found to exist prior to the start of systematic instruction. Only a small amount of evidence is to be had from previously reported research; the more complete and convincing evidence comes from an original study by the junior author, reported here for the first time. Evidence on readiness obtained during the course of learning studies will then be summarized in Chapter IX, most of it previously published, but some of it from the new studies reported in Part II of this monograph.[3]

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[3] It is not the purpose of Part III to determine whether the multiplication combinations *should* be taught in Grade III (all or part of them), or whether they should be postponed to a later point in the grades. Ability to learn is only one criterion for grade placement. It may be justifiable to postpone topics long after children *could* learn them on the basis of other criteria. For example, the child may not *need* the topic (can get along without it) at the point where he can learn it. Or, we may prefer to postpone a learnable topic in order to do a more thorough and more extensive job with other topics.

CHAPTER VIII

EVIDENCE ON READINESS, PRIOR TO INSTRUCTION

For convenience this chapter is divided into two sections. In the first section will be reviewed several earlier investigations, all except one of which have been published and none of which was concerned primarily with the question of readiness for the multiplication combinations. Three studies in this group are inventories of children's number knowledge before the start of systematic instruction in arithmetic, and none gives more than passing attention to multiplication. Four other studies report knowledge of the combinations immediately prior to instruction thereon. The second and larger section of the chapter will be devoted to a new investigation which was designed specifically to discover children's readiness for the combinations in Grade III.

PREVIOUS RESEARCH

Inventories

Grant (9)[1] administered Test 5, Numbers, of the Metropolitan Readiness Tests, to 563 children just beginning Grade I in nine Cincinnati schools. One item in the test called for the use of 3 × 2 in a verbal problem, the answer to be indicated by encircling the correct number in the printed series 9, 7, 10, 6, and 15. Only 0.4 per cent of his 145 subjects with IQ's below 90 scored successes, and the corresponding per cents for the 252 subjects with IQ's between 90 and 109 and for the 166 subjects with IQ's of 110 and above were 18 and 31, respectively. The correlation between success and IQ is what would be expected; but Grant's few data, as he himself would insist, reveal little about his subjects' knowledge of the multiplication combinations—or even about their knowledge of the one fact 3 × 2 = 6. "Problems" are usually harder for children than are abstract examples, and the method of recording answers introduced irrelevant difficulty.

In an earlier study (29) Woody had tests given individually to 2,895 children in thirty-nine school systems. His subjects were enrolled in five half-grades, from kindergarten through Grade IIA, but the tests were always administered in the half-grade immediately before that in which systematic instruction was started. Three test items involved the identification of successive groups numbering 3, 5, and 4. The procedure was to have children mark off each equal group as specified, in printed rows of stars. Only the data for 1,897 Grade IA children were reported: 90 per cent were successful with groups of 3, 79 per cent with groups of 4, and 76 per cent with groups of 5.

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[1] Numbers of this kind refer to the bibliography at the end of Chapter II, pp. 40-42.

The Woody data are only indirectly significant for the problem at hand. His subjects were not required to announce the totals obtained by their grouping, and hence the test situation did not involve the process of multiplication as such. But it did involve an ability basic to multiplication, namely, the ability to deal with series of groups of equal size. The plan for his investigation called for observations of the methods of work employed by subjects in identifying groups; but his interviewers apparently supplied few usable returns, and Woody inferred that, though his subjects could identify groups, they did so only by the most immature of methods, chiefly counting.

With this conclusion Carper[2] takes issue. Her subjects were 270 children in three North Carolina cities who had been in Grade I less than three weeks when tested. Their mean CA was 77.6 (S.D., 4.5), and their mean MA, 77.9 (S.D., 9.4). One part of her test consisted in having children draw "boxes" around groups of printed objects, the objects in each row being of the same size and equally spaced with respect to each other. Specifically, they were to mark (a) groups of two rabbits in a row of eleven, (b) groups of three boats in a row of eleven, (c) groups of four dogs in a row of eight, and (d) groups of five trees in a row of twenty-two. As in Woody's experiment, it was not required to find the total in each row.

Unlike Woody's interviewers, Carper, who tested all her subjects herself, noted carefully the means taken by each child to divide the rows of objects. She isolated three main procedures—grouping, partial counting, and counting—and was able to identify each by objective criteria. Her principal table (Table 4, p. 52) is reproduced below in modified form:

The per cents of successful grouping are especially high for the

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[2] Doris V. Carper, "A Study of Some Aspects of Children's Number Knowledge Prior to Instruction." Unpublished Doctor's dissertation, Duke University, 1941. 162 pp.

TABLE 31

REPRODUCING GROUPS BY DIVIDING ROWS OF PICTORIAL OBJECTS

Method of Procedure	*Per Cent of Correct Responses, by Number*			
	2	*3*	*4*	*5*
Grouping	95.2	83.7	38.5	1.5
Counting	0.7	8.2	42.9	57.7
Partial counting	....	2.2	5.9	1.9
*Total**	95.9	94.1	87.3	61.1

*The per cents of incorrect responses were, in the order above, 4.1, 6.0, 12.6, and 38.9.

numbers 2 and 3, in spite of Carper's very high standards. (If counting was used in obtaining a single group in a row, no credit for grouping was allowed for that row. Furthermore, an error was entered for a row if any single group contained the wrong number of objects, or if any object was skipped.) Her high standards probably affected adversely the grouping into 4's and 5's especially, as did also the character of the objective material used for these numbers. For these reasons the per cents of grouping for 4 and 5 are to be regarded as minimal. Nevertheless, with all these limitations, which are mentioned by Carper herself, these children clearly manifested surprising ability to deal effectively with successive groups of two and of three objects each and some skill in dealing with groups of four and of five objects. *To this extent,* then, it may be said that they were ready for multiplication, even at the start of Grade I.

Knowledge Immediately Prior to Instruction

So much for the inventories which relate to the readiness of children for multiplication as much as two and a half years before they are taught the combinations; next to be considered are four learning studies. In these studies are reported data on the knowledge of the combinations at the start of systematic instruction thereon.

Preceding his experiment on the value of self-directed activities in learning the combinations, Fowlkes (8) tested his third-grade subjects. The average child wrote correct answers for twenty-three of the one hundred combinations, and the range was 0-66. In similar circumstances Norem and Knight (14) gave two oral tests to each of their twenty-five third-grade subjects. The average child gave correct answers for thirty-two combinations on both tests, and the range was from 1 (a very young child, aged eight years, one month) to 65 (a child aged eight years, eleven months). The agreement between the Norem-Knight and the Fowlkes data is noteworthy.

Some years later Dahlgren under Brueckner's direction investigated the factor of transfer in learning the multiplication combinations (4). On the pretest 23 per cent of the answers for the nine facts with 2 as multiplier were correct, and 25 per cent for the facts with 2 as multiplicand. The per cents on the pretests for 3 and for 3's were 51 and 52, respectively, and for 4 and for 4's, 54 and 52, respectively. The per cents for the later tests are spuriously high because of some overlapping of test content. Nevertheless, the per cents of knowledge prior to instruction are high, even though about half of the experimental subjects were drawn from Grade IIB, and so had the disadvantage of a full year's less experience with number.

The fourth and most recent investigation in this group was made by Wheeler (27). On the pretest in his study the median score of his third-grade subjects was 9.3, with Q_1 at 1.6, and Q_3 at 28.2; the range was 0 to 75. In this study alone, among the four here reviewed, the showing of the subjects was what might be called "poor," and there seems to have been little readiness for the combinations. If Wheeler's subjects in the two preceding grades had been taught arithmetic with as much emphasis upon drill and as little emphasis upon meaning as characterized their instruction in the course of the experiment that followed, their showing on the pretest is entirely understandable.

As a whole, the data in the four foregoing studies seem to mean that children by the middle of Grade III are ready for the multiplication combinations: that is, they are intellectually able to learn them. Under conditions of learning which were not conducive to learning, the experimental subjects actually learned a number of the combinations. The assumption is that under conditions designed to encourage learning they could learn many more. Admittedly, the "learning" which these experimental subjects exhibited has been inadequately described. About all that is certain about their "knowledge" is that they could furnish correct answers for combinations. How they obtained their answers, no one can say. From the data presented in Part II it is reasonably certain that many of the answers, even the correct answers, were secured by procedures far short of meaningful recall. Nevertheless, there is no denying the fact that many of the children, prior to instruction, already knew some way or ways to go about the task of getting answers. And this knowledge, however imperfect, argues readiness for systematic instruction.

A NEW STUDY OF READINESS

All the evidence with regard to readiness for the combinations thus far presented has been indirect, for none of the investigations reviewed was organized with specific reference to readiness. Indeed, there is but one published report of such an investigation, a learning study which will be considered in Chapter IX. But the following pages contain new data which relate directly to the problem of readiness, for the research was designed with this single question in mind.[3]

Preliminary Investigation

A preliminary investigation with fifty-four third-grade children was undertaken about a month before the main study, primarily to work out the technique for the latter. Both a group test and interviews were used. Results from the former indicated that the group test could be lengthened and made somewhat more difficult. Results from the interview were serviceable in planning the questions to be asked and in identifying categories for classification. The data obtained in the preliminary study, having been secured from a slightly superior group of children and by methods not exactly comparable with those used later, are presented at only one point in this chapter.

Subjects of the Main Investigation

In the main study the subjects were ninety-eight pupils in three schools in as many different small cities. All children took the group test early in February. For the interviews which followed at once, the children from each class were taken alphabetically, and it was possible to hold conferences with eighty-five. As a group, the experimental subjects were slightly below average in arithmetic ability, as evidenced by the fact that their classes were the only ones in their systems in which the multiplication combinations had not yet been presented. Accordingly, the results obtained with respect to readiness may safely be regarded as somewhat less favorable than may be expected with typical third-grade classes.

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[3] This study was planned and made by Doris V. Carper, and credit for the data reported belongs exclusively to her.

The Group Test

The mimeographed group test was administered without time limit. At the top of the test blank were forty abstract multiplication combinations; at the bottom, twelve spaces in which the children were to enter their responses to twelve flashcards, presenting as many facts by means of groups of concrete objects. Since the findings from the flashcards revealed nothing of special importance, only the data for the abstract combinations are reported.

Twenty-four different combinations were used, sixteen of them twice. These sixteen facts appeared, along with eight other facts, at the beginning of the test in mixed order, and at the end of the test in tabular order. Table 33 shows the facts used as well as their order in the test. The children were not instructed how to read the examples (all were in the form, three 2's =, the multiplication sign, ×, not being used). They were, however, told that they were about to try a new kind of arithmetic and that if they could not find answers for some examples, they were to omit them. None of the children seemed baffled by the unfamiliar task. On the contrary, they set to work with confidence. The slowest child finished in approximately twenty minutes.

Distribution of Scores on the Group Test

Since each correct answer was credited with one point, a perfect score was 40. Table 32 contains the distribution of scores. The mean was 29.9, with a standard deviation of 5.3, and the range was 2 to 40. The score of 2 was made by twelve children, all of whom made the same mistake of adding multipliers and multiplicands. If these twelve children are omitted,[4] the mean score becomes 32.0, with a standard deviation of 5.3; and this mean score represents 79.9 per cent of the possible score.

It will be noted (Table 33) that eleven of the forty combinations involve 10 as multiplier or multiplicand. If these eleven combinations are excluded, leaving twenty-nine combinations which are regularly taught as part of the so-called one hundred combinations, the average score for the eighty-six children is 81 per cent of the possible score.

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[4] It is not unreasonable to omit these cases from all tabulations except Table 32. After all, as the interviews later revealed, they were confused by the unfamiliar *form* of the examples, so confused that they could not have answered correctly. In other words, their errors on the group test do not signify unreadiness for the combinations, but simply inability to determine from the form of the examples what they were to do.

TABLE 32

Distribution of Scores on the Group Test

Number of Combinations Correctly Answered	*Number of Pupils Making Score*
40	2
39	3
38	7
37	5
36	10
35	9
34	6
33	4
32	4
31	3
30	5
29	4
28	1
27	3
26	6
25	2
24	2
23	4
22	1
21	2
20	1
15	1
12	1
2	12
Possible Score	40
Mean Score	29.9; S.D., 5.3
Median Score	32

TABLE 33

Comparison of Per Cents of Accuracy for Isolated and for Tabular Presentations

Combination	*Per Cent of Accuracy*		*Combination*	*Per Cent of Accuracy*	
	Isolated Order	*Tabular Order*		*Isolated Order*	*Tabular Order*
Ten 3's	32.5	37.2	Two 4's	96.5	95.2
Ten 2's	55.8	56.9	Two 10's	97.7	96.5
Five 3's	59.3	46.5	Two 5's	98.8	97.7
Three [illegible]'s	66.2	88.7	Two 2's	100.0	98.8
Six 2's	68.2	63.9	Seven 2's	....	53.8
Four 3's	73.2	67.4	Nine 2's	....	55.8
Five 10's	74.4	90.6	Eight 2's	....	59.3
Three 10's	82.5	90.6	Four 4's	....	73.2
Two 3's	87.2	94.3	Three 4's	....	80.2
Three 5's	91.8	89.5	Five 2's	....	81.3
Three 2's	93.0	83.7	Four 10's	....	94.3
Four 2's	94.3	86.0	Two 1's	....	100.0

All the children answered correctly for two 1's and for two 2's, and between 92 per cent and 99 per cent answered correctly for two 5's, two 4's, four 2's, three 2's, and three 5's. (These figures are for the facts when presented in isolated form.)

The data from the group test support the findings of Fowlkes, Norem and Knight, and Brueckner (see pp. 111-112, above) in showing a considerable degree of readiness for the multiplication combinations. It is highly probable that with the start these subjects had they profited considerably from systematic instruction which followed at once.

Analysis of Errors on Group Test

The papers of the eighty-six subjects who scored 12 or higher on the group test were examined in an attempt to discover why errors occurred. The incorrect responses were classified in five categories, and then each category was analyzed.

(1) *Isolated versus tabular presentation.*—As has already been explained, sixteen combinations appeared twice in the test, first in mixed or random or isolated form, later at the end of the test in tabular form. According to the latter grouping, all facts with 2 as multiplicand were placed in order: two 2's, three 2's, four 2's, etc., to ten 2's; then the facts with 3 as multiplicand, with 4 as multiplicand, and so on. It seemed possible that the form of presentation might have been one factor in determining success or failure in finding answers.

Such, however, seems not to have been the case with the children in the main investigation. There is marked inconsistency in the relative difficulty of the facts when presented in the two forms. For six facts (see Table 33), namely, ten 3's, ten 2's, three 3's, five 10's, three 10's, and two 3's—for these six facts higher degrees of success were scored in the tabular form; for the other ten facts, in the isolated form. Moreover, the coefficient of correlation (rank order) between the scores made on the sixteen facts in the two orders is .86. While this coefficient is not high in view of the close proximity of the two forms in the test, nevertheless, it is high enough to mean that each fact was of about the same relative difficulty regardless of form of presentation.

The findings in the preliminary investigation were quite different. The fifty-four children in this study were, it will be recalled, somewhat superior in arithmetic ability. In their group test seven combinations were presented in random order, and then in tabular order. For only one combination, two 5's, was the per cent higher for the isolated form (88.8 per cent as compared with 83.3 per cent) ; for all other six combinations—three 2's, three 3's, two 10's, five 10's, two 3's, and six 2's—the per cents were higher for the tabular presentation, in five cases by differences of nine points or more.

The dissimilarity in the two sets of results seems to be explicable in terms of the differences in quality of the two groups of subjects. The superior children in the preliminary investigation made use of the relationships implicit in the tabular presentations, and tended to increase the products in each series by the amount of the multiplicand (two 2's = 4; three 2's = 4, and 2 more, or 6; four 2's = 6, and 2 more, or 8; and so on). It is noteworthy that these children discovered these relationships for themselves, and that they did so from a single experience with grouped combinations. On the other hand, the slower children in the main investigation did not observe this relationship; or, if they observed it, they failed to make effective use of it. They were to this extent, then, less ready for the combinations than were the first group of children.

If the foregoing analysis is valid, it is significant for the teaching of the combinations. All children will not profit equally from tables, for some may fail to note the principles of relationship implicit in the tables. Children of average and of less than average ability may need to have their attention called specifically to the basis of organization. Otherwise, they will tend to learn the combinations in the tables just as they would learn them in random order.

(2) *Use of an incorrect sequence.*—Some children noted the presence of a sequence in the facts grouped according to their multiplicands, but failed to detect the nature of that sequence. For example, with the group of facts for 3's the responses were: "Two 3's are 6," "Three 3's are 7," "Four 3's are 8," and so on. Instead of adding the amount of the multiplicand with each new fact, these children added 1.

This "sequence of 1" error was quite common: twenty-one of the eighty-six subjects (and so, nearly one fourth) made this error on tabular combinations. The extent of this error is indicated in Table 34, which shows that it was responsible for between three tenths and six tenths of all wrong answers for these combinations. It was the frequent occurrence of this single error which kept the combinations

TABLE 34

Per Cents of Error Resulting from Use of a Sequence of 1

Multiplication Fact (in tabular form)	*Total Number of Errors on Combination (N=86)*	*Number of Errors Due to Sequence of 1*	*Per Cents of Errors Due to Sequence of 1*
Five 2's	16	5	31.2
Six 2's	31	15	48.3
Seven 2's	40	20	50.0
Eight 2's	35	18	51.4
Nine 2's	38	21	55.2
Ten 2's	37	21	56.8
Three 3's	10	6	60.0
Four 3's	28	12	42.8
Five 3's	46	18	39.1

in tabular form from enjoying consistent advantage over the mixed combinations in the group test.

The "sequence of 1" error is an interesting one. Its users clearly discriminated between the new form of the combinations and the forms in which addition and subtraction combinations are written; and so, they did not add the factors or subtract one from the other. Moreover, they recognized the existence of some kind of sequential organization; but, through inadequate analysis, they misinterpreted its character and perhaps reverted to the familiar practice of adding 1 each time, as in counting by 1's. In fact, in many cases the recognition of sequence was so strong as to lead to errors on grouped combinations which had been correctly answered in the mixed or isolated form. It is not claiming too much to insist that this independent discovery of the fact of sequence, even of a wrong sequence, is really evidence that these particular children were ready for the multiplication combinations.

(3) *Order of factors.*—Other investigators have called attention to the fact that one source of difficulty in the combinations may be that of the order of the factors—whether the first factor is larger or smaller than the second. ~~(See p. 34, above.)~~ The evidence on this point has not always been consistent, but it has all been obtained from children after they have been taught the combinations. The inconsistency in the research findings may well be associated with differences in instructional practices. For children who have been taught (or have themselves discovered) the principle of reversals, it should make little difference whether the multiplier is larger than the multiplicand or vice versa.

In the present study, which employed as subjects children who had not yet been taught the combinations, the relative magnitude of multiplier and multiplicand was an important condition of error or difficulty. According to Table 35, in every one of the seven pairs of facts which were presented in both direct and reverse forms in the group test the per cents of accuracy are higher when the multiplier is smaller. In other words, the number of times a certain unit is to be dealt with contributed more to difficulty than did the absolute size of that unit. The fact that these children did not discover for themselves the principle of reversals (namely, that the order of factors does not affect the product) implies that specific instruction is needed at this point.

TABLE 35

COMPARISON OF PER CENTS OF ACCURACY FOR DIRECT AND REVERSE FACTS

Combinations, in Direct Order	*Per Cents of Accuracy**	
	Direct Order	*Reverse Order*
Two 5's	98.2	81.2
Two 10's	97.1	56.3
Two 4's	95.8	90.1
Three 4's	80.2	70.3
Three 5's	90.6	54.0
Two 3's	90.7	88.3
Three 10's	86.5	34.8

*Averages for tabular and isolated presentations.

(4) *Addition of multiplier and multiplication.*—As has already been noted, twelve children consistently added the factors in the combinations; e.g., "Three 2's are 5," "Four 2's are 6," etc. This error completely disappeared when the combinations were presented orally to these children in the interviews. On this account the error has been attributed to the written form of the combinations, rather than to lack of readiness for multiplication as such; and the children who made this error have been omitted from all tabulations except that in Table 32.

(5) *Size of product.*—It has been suggested many times (see p. 34, above) that size of product is directly related to the relative difficulty of combinations, in such a way that more errors are made when products are large than when they are small. In a real sense, this observation, if valid, is significant for the problem of readiness for the combinations. It is not impossible that children are ready to learn combinations with small products before they are ready for

combinations with large products. If so, it is inaccurate to speak of children's being ready or not ready "for the combinations," in an all-or-none sense. Rather, it may be necessary to think of them as being ready for these, but not for those, combinations.

In the case of the combination "three 2's," an answer of "5" is one thing, and an answer of "69" is something else again. The answer "5" betokens a familiarity with the relative size of numbers and with what multiplication does to factors which is quite lacking in the answer "69." Recognition of such facts as the foregoing led to a somewhat unusual method of analyzing incorrect answers on the group tests. The disparities between written products and true products were expressed in terms of percentage, in order to preserve measures of their "reasonableness." A sample of the table of values used in the tabulation follows:

True Product	Per Cents by Which the Given Written Answers Deviated from the True Products			
	15	30	50	Over 50
6	5:7	4:8	3:9	2:10
10	8:12	7:13	5:15	4:16
15	13:17	10:20	8:23	7:26
20	17:23	14:26	10:30	9:34

For two 3's and for three 2's, answers of "5" and "7" deviate approximately 15 per cent from the true answer 6, while answers of "4" and "8" deviate approximately 30 per cent, and so on. The arbitrariness of the limits set is obvious, as are also the resulting inaccuracies. Nevertheless, the procedure is sufficiently refined for the purpose at hand, which is to note the relation of size of product both to frequency of error and to amount of that error.

The data yielded by this analysis are assembled in Table 36. No errors at all were made when the product was 2 (the combination two 1's occurred but once for each of the 86 subjects). The product 4 should have been written 172 times, once each by the 86 subjects for the mixed and for the tabular forms of two 2's. One error appeared on the test papers, and this deviated from the true answer 4 by 50 per cent.

The first fact revealed by the tabulation is that the number of errors increased with size of product. For products of 9 and less the per cent of error was 8.9 (99 ÷ 1118); for products 12 to 18 inclusive, 29.6; and for the four products in the last group, 21.3. (The figure for the last group is somewhat out of line, but the discrepancy

TABLE 36
Relation of Errors to Size of Product

Product*	*Number of*		*Number of Errors Deviating*			
	Occurrences	*Errors*	*15%*	*30%*	*50%*	*Over 50%*
2	86	0	..	..	..	..
4	172	1	..	..	1	..
6	344	36	4	6	23	3
8	344	23	4	9	6	4
9	172	39	4	35	..	..
Total	1,118	99	12	50	30	7
12	430	118	57	25	28	8
14	86	37	25	8	3	1
15	344	87	30	39	7	11
16	172	55	23	15	14	3
18	86	34	19	11	4	..
Total	1,118	331	154	98	56	23
20	344	69	19	23	18	9
30	344	100	37	10	32	21
40	86	5	1	..	4	..
50	172	27	6	3	9	9
Total	946	201	63	36	63	39
Grand Totals	3,182	631	229	184	149	69
Per cent of Total Errors	.....	...	36	29	24	11

*The figures for the product 10, omitted here to separate more widely the first two groups of factors, are, from left to right as above: 258, 19, 11, 2, 4, 2.

is of little importance, since most of the combinations in this group involve 10 as multiplier or multiplicand.)

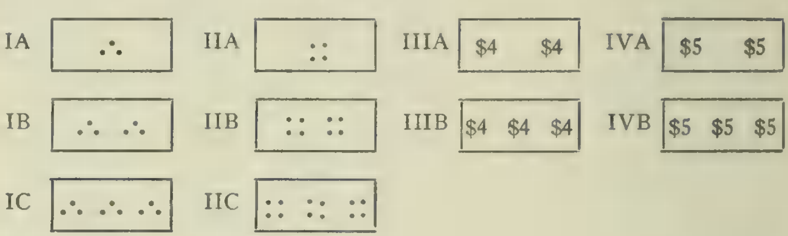
The second fact is that the errors deviated less with the smaller products. For the first group, 3.3 per cent of the errors deviated by 50 per cent or more, while the corresponding per cents for the last two goups are 7.1 and 50.8, respectively. Some of the significance of this conclusion is lost because the possibilities of error by underestimation are far fewer with the smaller products than with the larger. Nevertheless, the probability remains that the children's greater familiarity with the smaller numbers—their greater understanding of these numbers—tended to protect them from gross errors. The implication for teaching the combinations is clear: at the start of instruction children are readier for combinations with small products than for combinations with large products.

Personal Interviews: Purpose, Materials, Procedure

If we are to know how children learn the multiplication combinations, we must know how they think of them and how they arrive at their answers. This point of view has been steadily emphasized throughout this monograph, and the value of data on children's processes has been amply demonstrated in Part II. In the investigation now being described, data on the subjects' processes were obtained by individual interviews held with eighty-five of the ninety-six children.

The test used in the interview included fifteen examples, arranged in the following order:

Concrete forms (presented on cards):



Abstract forms (presented orally):

Two 2's =	Two 4's =	Four 2's =
Three 3's =	Three 5's =	

As soon as a card with a concrete form was presented, these questions were asked:

(1) How many dots (or dollars) are on this card?
(2) What did you do (think) to find out?
(3) What other ways could you find out?

The abstract combinations were presented orally; for example, for the first the question, "How many are two 2's?" was asked, followed by questions (2) and (3) above.

For each combination, whether presented concretely by card or orally, the child's answers were recorded on an Interview Blank. The records then were used in the tabulations about to be reported. Only the responses to question (2) are considered here, those for questions (1) and (3) being omitted for several reasons. In the first place, it was immaterial to the present purpose whether the subjects obtained correct answers for the combinations (as a matter of fact, most answers were correct), for data with respect to accuracy of response had been secured on the group tests. In the second place, the answers to question (2) contained precisely the information the interviews were set up to secure. In the third place, the answers to question (3) revealed principally that the children were familiar with a number of immature processes, such as counting by 1's, which however they had already replaced with more mature procedures.

Types of Process Employed

The processes which the children used to obtain their numerical answers (i.e., the processes which the children said they used or which they were observed to use in case explanations were incomplete) were classified according to twelve categories. The categories and the symbols used to represent them are given below, each with an illustration; but there are so many and the code letters are so ambiguous that the reader may experience some difficulty in remembering them. It may help to know the general rationale of the symbols.

It is obvious that the children could deal either with the basic groups (such as $4 in Card IIIA or .˙. in Card IC), or with smaller groups, subgroups larger than 1, within the basic group. The letter B in the code symbols stands for "basic group," and the letter S for "subgroup." But the basic group could be responded to in various ways. For example, some multiple of the basic group might be used and supplemented in some way or other. The symbol B_n stands for this sort of practice, so far as the unit is concerned. Or, a single basic unit might be supplemented in some way. A basic unit so used has the symbol B_1.

So much for the units which were used to obtain answers; next comes the question of what was done with them. They might be used in multiplication (M) or in addition (A), or be immediately recognized (R), or be counted (C). Somewhat similar procedures could be employed with subgroups (S) or with a multiple of the basic unit (B_n) or with a single basic unit (B_1). Hence, the letters M, A, and C will be found associated with these symbols as well as with B.

Card IIC (:: :: ::) and the abstract combination two 5's are used in the following illustrations of processes:

1. Recognition and Use of Successive Basic Groups or Multiplicands (B)
   BR—Immediate recognition of the total product
      Card IIC—Saying "12" immediately. ("Just knew it," or "Just saw it.")
      Two 5's—Saying "10" immediately
   BA—Additive use of basic group
      Card IIC—Saying, "4 and 4 are 8, and 4 more are 12."
      Two 5's—Saying, "5 and 5 are 10."

BC—Counting in terms of basic group
    Card IIC—Saying "4, 8, 12."
    Two 5's—Saying "Two 5's are 10."

2. Supplementation of a Multiple of the Basic Group (B_n)
   B_nA—By addition (used with only three of the basic groups)
       Card IIC—Saying, "8 and 4 are 12."
   B_nC—By counting (used with only three of the basic groups)
       Card IIC—Saying "8, 9, 10, 11, 12."

3. Use of Subgroups (S) (Used only with concrete groups, cards)
   SA—Additive use of equal subgroups
       Card IIC—Saying, "2 and 2 are 4; and 2 and 2 are 4; that makes 8; 2 more makes 10, and 2 more makes 12."
   SM—Multiplicative use of equal subgroups
       Card IIC—Saying, "Six 2's are 12."
   SC—Counting in terms of equal subgroups
       Card IIC—Saying, "2, 4, 6, 8, 10, 12."
   S_uA—Additive use of unequal subgroups
       Card IIC—Saying, "7 and 5 are 12."

4. Supplementation of a Single Basic Group (B_1)
   B_1C—By counting
       Card IIC—Saying, "4, 5, 6, 7, 8, 9, 10, 11, 12."
       Two 5's—Saying, "5, 6, 7, 8, 9, 10."
   B_1A—By addition
       Card IIC—Saying, "4 and 2 are 6, and 2 are 8, and 4 are 12."
       Two 5's—Saying, "5 and 2 are 7, and 3 makes 10."

Interview Findings

Extent of use of processes.—The principal conclusion to be drawn from the interview data has to do with the relative use made of the various processes, and, more especially, with the extent to which multiplication processes were used. The facts are assembled in Table 37. Let the reader bear in mind that the symbol B stands for basic groups used repeatedly, B_n for some multiple of the basic group, B_1 for a single basic group, S for subgroups, A for addition, C for counting, M for multiplication, and R for immediate recognition.

Of the twelve processes, those designated as BC, BM, B_nA, B_nC, and SM clearly involve understanding and use of multiplication. To these processes should be added BR (immediate recognition of the total) which was found to predominate when only one basic group was presented on a card. These six processes were responsible for 833, or 65.3 per cent, of the 1,275 responses in the interviews. In other words, nearly two thirds of the combinations presented were solved by processes closely allied to multiplication. This finding testi-

TABLE 37

PERCENTAGE CLASSIFICATION OF PROCESSES USED BY EIGHTY-FIVE SUBJECTS IN INTERVIEWS

Interview Item	*Per Cents of Responses According to Process Used*											
	*BR**	*BA*	*BC**	*BM**	B_nA*	B_nC*	*SA*	*SM*	*SC*	S_uA	B_1C	B_1A
Card IA	85.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.5	10.6	0.0	1.2
IB	3.5	53.1	4.7	30.7	0.0	0.0	2.4	0.0	2.4	1.2	0.0	2.4
IC	2.4	8.8	14.2	29.5	37.8	3.5	0.0	0.0	4.7	0.0	0.0	0.0
IIA	51.9	0.0	0.0	0.0	0.0	0.0	35.4	2.4	10.6	0.0	0.0	0.0
IIB	4.7	50.7	0.0	37.8	0.0	0.0	2.4	0.0	3.5	0.0	0.0	1.2
IIC	1.2	9.4	5.9	20.1	38.9	13.0	2.4	0.0	5.9	0.0	0.0	3.5
IIIA	3.5	61.4	1.2	33.0	0.0	0.0	0.0	0.0	1.2	0.0	0.0	0.0
IIIB	3.5	8.3	8.3	31.9	37.8	5.9	1.2	0.0	1.2	1.2	1.2	1.2
IVA	3.5	57.8	10.6	26.0	0.0	0.0	0.0	0.0	2.4	0.0	0.0	0.0
IVB	1.2	11.8	15.3	31.9	36.6	1.2	0.0	0.0	0.0	0.0	2.4	0.0
Two 2's	30.7	43.7	5.9	18.9	0.0	0.0	0.0	0.0	0.0	1.2	0.0	0.0
Three 3's	2.4	16.5	11.8	18.9	41.3	3.5	0.0	0.0	2.4	2.4	1.2	0.0
Two 4's	16.5	53.1	8.3	17.7	1.2	0.0	0.0	0.0	2.4	1.2	0.0	0.0
Three 5's	9.4	3.5	16.5	28.3	37.8	1.2	0.0	0.0	1.2	1.2	1.2	0.0
Four 2's	4.7	16.5	26.0	26.0	17.7	0.0	0.0	0.0	2.4	7.1	0.0	0.0
Average	14.9	26.3	8.5	23.3	16.6	1.9	2.9	0.2	2.9	1.7	0.4	0.6

*Processes using the multiplication processes.

fies convincingly to the readiness of these subjects for systematic instruction on the combinations.

Effect of form of presentation.—That these subjects had progressed a considerable distance toward abstractness in their thinking of the numbers represented in the combinations is revealed by another tabulation (Table 38). It made little difference to these children whether the numbers were presented as digits, as grouped dots, or as digits in association with the dollar sign. Six comparisons are possible within the interview data, and the largest difference in per cent of responses involving multiplication processes is about 10 points (for Card IC, .·. .·. .·., and the abstract combination three 3's).

Effect of number of groups (size of multiplier).—A very striking difference is found in the number of responses involving multiplication when the combinations contained *two* as compared with *three* basic groups. From the data in Table 39 it is evident that when three basic groups were used, approximately twice as many processes made use of multiplication as when but two such groups were presented. In the latter case these subjects tended to add.

This difference in process is understandable. The multiplication process is noticeably superior to addition only when at least three

TABLE 38

Relative Extent to Which the Multiplication Process Is Used When the Facts Appear in Different Forms

Form of Presentation	*Per Cents of Responses Involving Multiplication Process*
Dots versus Dollars:	
Card IIB (: : : :)	42.5
Card IIIA ($4 $4)	37.7
Card IIC (: : : : : :)	79.1
Card IIIB ($4 $4 $4)	87.4
Dots versus Abstract Digits:	
Card IC (.·. .·. .·.)	87.4
Three 3's	77.9
Card IIB (: : : :)	42.5
Two 4's	43.7
Dollars versus Abstract Digits:	
Card IIIA ($4 $4)	37.7
Two 4's	43.7
Card IVB ($5 $5 $5)	86.2
Three 5's	93.2

TABLE 39

Relative Extent to Which the Multiplication Process Was Used for Combinations Involving Two and Three Basic Groups

Card, and Combination	*Per Cents of Responses Involving Multiplication Process*
Card IB (.·. .·.)	38.9
Card IC (.·. .·. .·.)	87.4
Card IIB (: : : :)	42.5
Card IIC (: : : : : :)	79.1
Card IIIA ($4 $4)	37.7
Card IIIB ($4 $4 $4)	87.4
Card IVA ($5 $5)	40.1
Card IVB ($5 $5 $5)	86.2

equal groups are involved. This amounts to saying that it is about as easy and quick to think "2 and 2 are 4" as it is to think "two 2's are 4." For this reason the ability of children to recognize the need for multiplication and to use the process effectively is better revealed by their reactions to three or more equal groups than by their reactions to two such groups. So far as the experimental subjects are concerned, it should be noted that over four fifths of the responses to three groups of 2's, 3's, 4's, and 5's made use of basic multiplication concepts and skills. This is but further evidence that these children were ready for instruction on the combinations.

Summary of the Investigation

1. A large group of children who were entering the second half of the third grade and who had had no instruction in multiplication were highly successful in responding to a group test containing forty multiplication combinations. The average score of the eighty-six children tested represented 78.8 per cent of the highest possible score.

2. Analyses of the errors on the group test revealed that:

a. For children who recognize the kind of sequence present in tables (each successive combination having a total larger by the amount of the multiplicand than the preceding combination), combinations produce a higher per cent of accuracy when grouped in tables than when presented in random order.

b. Not all children make this generalization themselves. Approximately one fourth of the subjects realized that some sort of sequence characterized the facts in tabular form, but failed to grasp the nature of that sequence. These children added 1 instead of the amount of the multiplicand to successive products. This response may have resulted from transfer in counting by 1's; or it may have been produced by some generalization, such as: since one unit is added to the first figure (multiplier) each time, one unit should be added to the product as well.

c. The direct form of stating the combinations (with the smaller number as multiplier) resulted in higher per cents of accuracy than did the reverse form. This observation rests upon comparisons for seven pairs of combinations. The generalization concerning reversals is apparently not readily hit upon without specific guidance and should be made the subject of direct instruction. That the generalization is not easy to discover is understood, if one compares the two concrete presentations below:

two 3's = 6 .·. .·.   three 2's = 6 : : :

The concrete forms do not *look* alike; the separate units have to be combined in different arrangements.

d. The size of the product affected accuracy of response. More errors were made with large products than with small, and the size of the error (its deviation from the true product) was greater with large products. Still, only a few more than a third of all errors deviated from the true products by as much as 15 per cent and about two thirds deviated by 30 per cent or less. These facts, together with those presented above, warrant the conclusion that these subjects were ready for the multiplication combinations.

3. The personal interviews disclosed the processes used by the children to find answers for fifteen combinations which were presented by means of abstract numbers, dollar numbers, and grouped dots. Of the 1,275 responses, 833 could be unquestionably classified as involving an understanding of multiplication and the ability to use the process. Four fifths of the processes could be so classified when the number of groups (multiplier) was 3 instead of 2. The different forms in which the combinations were presented had little effect upon the process used, from which fact it may be concluded that most of these children were able to approach the study of multiplication on an abstract basis.

CONCLUDING STATEMENT

The findings in this new study, just summarized, agree with the results of earlier investigations in which subjects have been employed who had had no instruction in multiplication. There seems to be little question on the basis of this evidence that children by the middle of Grade III are ready for multiplication combinations, or, at least, for some of the multiplication combinations, probably those whose products are small. Whether or not this condition of readiness means that the combinations, all or part of them, *should* be taught in Grade III is another matter. Ability to learn is only one of several considerations which determine when a topic should be taught.[5] But this much can be said with considerable certainty: postponement of instruction on the combinations to Grade IV or later cannot be justified on the ground that children in Grade III are incapable of learning them.

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[5] Ben A. Sueltz, "Arithmetic Readiness and Curriculum Construction," *Mathematics Teacher*, XXX (Oct., 1937), 290-292.

CHAPTER IX

EVIDENCE ON READINESS FROM LEARNING STUDIES

So far as a survey of research literature reveals, there has been but one learning study which was specifically designed as an attack upon the problem of readiness for the multiplication combinations. This investigation, by the Committee of Seven (24),[1] is given first place in this chapter. The second part of the chapter will be devoted to data relating to readiness which are available in other published reports and in the investigations described in Part II of this monograph.

THE COMMITTEE OF SEVEN INVESTIGATION[2]

The study of readiness for the multiplication combinations is but one of several similar studies made on arithmetic topics by the Committee of Seven. The reports of results have seldom carried many of the details of technique which characterized experimentation on specific topics. This is true in the case of the study of readiness for the multiplication combinations. It is fair to assume, however, that in this instance as in others the Committee's general plan of investigation was followed.

The Technique Employed

The Committee first ascertained by circulating questionnaires the usual grade placement of a given arithmetic topic (here the multiplication combinations). They next arranged to have the topic taught in a number of schools in the usual grade and in the grades immediately preceding and following that grade. (In the case of the multiplication combinations these grades were supposedly Grades II, III, and IV.) Tests were then administered to the experimental subjects—an intelligence test to secure measures of MA and a "foundations test" for each topic. (In the present instance the latter was a test on the one hundred addition facts.) The Committee supplied to each co-operating school definite directions and materials covering content, length of experimental period and daily class, methods of teaching, tests, and the like. (In the investigation with which we are concerned, the content was the eighty-one combinations without 0, and the experimental period lasted six weeks, twenty minutes a day. The methods of teaching are not described, but they probably were largely those of drill.) Tests on the topic were administered at intervals during the experimental period, at the conclusion of the same, and six weeks later (the retention test).

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[1] Such numbers as this refer to the bibliography at the end of Chapter II, pp. 40-42.

[2] This committee was composed of seven principals and supervisors in several cities of northern Illinois, with Carleton W. Washburne, of Winnetka, as chairman. Since the reports of the Committee have usually been made by the chairman, the terms "Committee of Seven" and "Washburne" will be used interchangeably. The data on the multiplication combinations are contained in the Committee's first account of its work, which appeared in 1930.

In working their data, the Committee first rejected all subjects who had failed to make some minimum score on the foundations test. (What this minimum score on the addition facts was, we are not told specifically. At one place (p. 656) Washburne advises the postponement of the multiplication combinations to a certain time "even in the case of children who make a score of 92 to 100 on the addition facts test." And on page 670, 96 per cent is mentioned as the "minimum" score for one standard.) For the subjects who were left, a comparison was made between retention test scores and MA's, and the arithmetic topic was assigned to that MA at which for the first time about 75 per cent made satisfactory scores (usually 80 per cent) on the retention test.

The Committee's Findings for the Multiplication Combinations

After rejecting all children who made unsatisfactory scores on the test on the addition facts (whatever the criterion was), the Committee had left 655 subjects, with MA's ranging from 7 years 0 months to 12 years 3 months. The data for these children are presented below in a slightly modified form:

	MENTAL AGE									
	7:0-7:5	7:6-7:9	7:10-8:3	8:4-8:9	8:10-9:3	9:4-9:7	9:8-10:1	10:2-10:7	10:8-11:3	11:4-12:3
Number of children ...	10	25	64	111	149	144	101	77	35	11
Per cent making 80 per cent on retention test	20	16	34	44	46	44	55	61	69	55

These data are read as follows: Ten children had MA's of 7 years 0 months to 7 years 5 months; of this number, 20 per cent made scores of 80 or better on the retention test on the multiplication combinations, this test being given six weeks after the conclusion of the experimental period.

It will be observed that the per cents of children who made satisfactory scores on the retention test mounted steadily with MA except at three points. The break at MA 7 years 6 months is probably spurious and may be attributed to the small samples of subjects. The break at the highest MA, where there is an actual drop, is referred to by Washburne as "a tendency to flatten," a phenomenon noted by the Committee in their other studies (the data being graphed in the form of curves). Washburne notes also the "flattening" at MA 8 years 4 months, and states (pp. 654-655): "Apparently, therefore, the mental age 8 years 4 months represents a definite level of mental growth before which there is rapidly increasing capacity to learn and after which the capacity to learn the multiplication facts increases less rapidly." When the retention test scores were plotted against the final test scores, the unsmoothed curve broke also at MA 8 years 4 months, with a second flattening, as before, at MA 10 years 2 months.

Washburne then points out that while children with a mental age of 8 years 4 months who know the addition combinations will retain 90 per cent of what they have learned over a period of six weeks, the trouble is that they actually have not much to remember. For they learn only forty-two multiplication combinations. He argues, next, that postponement to a mental age of 10 years 2 months would be much better, for then children will know more of the combinations. The Committee in 1930 recognized the need for the multiplication combinations in other arithmetic work, and hence hesitated to recommend postponement to 10 years 2 months. "This much, at least, can be said with certainty: Multiplication facts should not be taught below a mental age level of 8 years 4 months, nor to children who have not attained virtual mastery of addition facts" (p. 656). In the conclusion of the report the Committee gives as the "minimal mental-age level" for the multiplication facts "8 yr. 4 mo. *or* 10 yr. 2 mo." The wide discrepancy between the two placements, nearly two years, should be noted.

Later Findings of the Committee of Seven

In their last report[3] the Committee present an analysis of test data on the multiplication combinations, whether the original data or new data, we do not know for certain. In this report the combinations are divided into the "easy" facts with products of 20 or less, and the "hard" facts with products of more than 20. Two sets of curves are presented, one set for the easy facts and another for the hard facts.

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[3] Carleton Washburne, "The Work of the Committee of Seven on Grade-Placement in Arithmetic," *Thirty-eighth Yearbook* (National Society for the

The data for the *easy* facts are plotted in four curves, the base line for which is marked off in terms of MA, the vertical axis in terms of the per cents of children making 80 per cent on the retention test. These data are sampled below:

	Per Cents* Making 80 Per Cent on the Retention Test, According to MA		
	7:9	8:9	10:3
(1) "*all* children making satisfactory retention tests scores, regardless of their 'foundations' and pretest scores"	75	80	95
(2) "all children with adequate foundations"	84	88	96
(3) "all children with low pretests"	70	75	85
(4) "all children with adequate foundations and low pretests"	75	79	88

* These per cents are estimated from the curves.

The last form of treatment (4) is the one now used by the Committee in recommending placement. (The inclusion of "pretests" in the criterion seems to mean that new data were used as the basis for this last report unless "pretests" means "foundations tests.")

The Committee's findings with regard to the *hard* facts are peculiar indeed: the children with "high foundations and high retention test scores" *never* learned these facts to satisfy the Committee's criterion for placement. Only about 24 per cent with MA's averaging 7 years 9 months secured scores of 80 per cent on the retention test, and this per cent never got higher than 55, even for children with MA's of 10 years 9 months. Fewer than 5 per cent of children with "high pretests" attained the Committee's standard of 75 per cent or 80 per cent on the retention test, regardless of MA (and the MA's ran as high as 10 years 9 months)! The Committee has nothing to say at this point about the placement of the hard facts.

Later on (p. 301), Washburne is not so reticent. His conclusions are quoted in full:

> The easy multiplication facts can be taught at this level (MA 8 years 9 months) very effectively. Multiplication facts fall, with few excep-

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Study of Education; Bloomington, Ill.: Public School Publishing Co., 1939), pp. 304-305, 310.

> tions, into two categories, those with products of 20 and less, and those with products of more than 20. The former belong definitely at this level; the latter about three years later. An exception is 5 x 5, which was correctly retained by 89 per cent of children at mental age 8. An exception in the other direction is 4 x 3 (with 3 written above the 4). This was correctly retained by only 68 per cent of 8-year-olds. In general, however, this is the obvious level for the multiplication facts with products of 20 and less, provided children have an adequate foundation in the addition facts.

The Committee's proposals, if adopted, would keep the easy facts about where they now are, late in Grade III, but it would defer the hard facts until Grade VI! Extraordinary as is this recommendation, even more extraordinary is the method by which Grade VI or MA 10 years 6 months was chosen for the last of the combinations. The Committee had no data for children in this grade, or apparently for children in Grade V; and the data for Grade IV children certainly warrant no such extrapolation as the Committee undertakes.

Criticism of the Committee of Seven Standards

A great deal of space has been given the Committee of Seven investigation (or investigations), primarily because the work of the Committee has been widely influential in all phases of arithmetic. Various members of the Committee, and especially its chairman, have propounded the "stepped-up" curriculum with a good deal of vigor. Courses of study and textbooks alike reflect acceptance of the Committee's recommendations, and arithmetical concepts and skills have been moved higher and higher in the grades. The research basis for this program of postponement lies almost wholly in the investigations of the Committee of Seven; yet, the findings of these investigations frequently go counter to those of other research which, in the writers' opinion, has been at least as competently done.

It is not at all difficult to criticize the experimentation of the Committee of Seven as a whole.[4] As a matter of fact, the Committee's chairman has conceded several experimental weaknesses, weaknesses critical enough, it would seem, to invalidate any research findings, but weaknesses which to the chairman are apparently unimportant.[5]

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[4] The senior author of this monograph has done so: William A. Brownell, "A Critique of the Committee of Seven's Investigations on the Grade Placement of Arithmetic Topics," *Elementary School Journal,* XXXVIII (March, 1938), 495-508. This criticism the chairman has answered: Carleton Washburne, "Reply to Brownell's Critique of the Committee of Seven Experiments," *Elementary School Journal,* XXXIX (Feb., 1939), 417-430.

But we are concerned here only with the Committee's research on the multiplication combinations. The following criticisms may be offered:

1. The Committee's original lower "minimal mental age" of 8 years 4 months was wholly unjustified in terms of their standard of mastery. At this mental age only 44 per cent of the children scored 80 per cent on the retention test.

2. The original higher "minimal mental age" of 10 years 4 months was likewise unwarranted by the Committee's data. At this mental age only 61 per cent (instead of 75 per cent) of the children satisfied the criterion, and children of still higher MA had an even poorer record.

3. In the revised 1939 proposals the Committee's placement of the easy facts at MA 8 years 9 months is wholly arbitrary. True, at this age 79 per cent of the children met their criterion, but if other criteria had been used (and they might as validly have been used), the placement would be much lower. For example, if the initial arithmetical measure had been "adequate foundations," the easy facts could be assigned to MA 7 years 9 months, or even lower. (No data for subjects with lower MA's are reported.) Moreover, the MA standard of 8 years 9 months for the *easy* facts is five months higher than the original (lower) standard for *all* facts.

4. The new placement of the hard facts at "about three years later" (and so approximately at MA 11 years 9 months) is totally without basis in the Committee's data, as has already been pointed out.

5. The Committee's subjects revealed an amazing lack of mastery. An average of forty-two of the eighty-one combinations taught in the original study were retained by children with MA's of 8 years 4 months; yet all these children (apparently) had good foundations in the addition combinations. And the failure of *any* subjects regardless of MA to learn and retain more than about the same per cent of the hard facts is difficult to explain, except according to the hypotheses advanced in the following paragraphs.

6. The validity of the Committee's standards rests, among other things, upon the validity of their measures of learning or retention.

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[5] Carleton Washburne, "The Values, Limitations, and Applications of the Findings of the Committee of Seven," *Journal of Educational Research,* XXIX (May, 1936), 694-707.

And in turn the validity of these measures rests in large part upon the way in which the tests were given. (The test content was obviously valid, consisting as it did of the eighty-one combinations which had been taught.) How is the low level of mastery shown by the Committee's subjects to be explained? One hypothesis is that the retention test was a speed test, with time limits adjusted so that only the fastest children could finish. In this case children of average and less than average ability had little opportunity to show what they really knew about the combinations. Because of the Committee's technique for determining mental age standards, the low level of apparent mastery inevitably raised the standards spuriously high, particularly in the case of the hard facts.

7. The Committee's standards are valid, if at all, only in schools which follow their program of instruction. That is to say, when children are to be taught the combinations in six weeks twenty minutes a day and by the Committee's methods of instruction—under these conditions children may be expected to be no readier for the combinations than were the Committee's subjects. Deviations from the Committee's specifications as to number of weeks of teaching, number of minutes per day, and methods of teaching—deviations at any of these points should make the Committee's standards inapplicable. At one time (**23**: 654) the chairman himself conceded that the short period of learning may have been prejudicial. It is common practice to allow a full school year to elapse between the presentation of the first combinations and the final mastery of all the combinations. The Committee allowed six weeks.

8. Occasionally stated explicitly and somewhat generally implicit in the Committee's reports is a conception of readiness which is highly questionable. It is evident that the Committee relies upon mere increase in age, the accumulation of birthdays, to produce the state of readiness requisite for economical learning. Experience as a factor in determining readiness is played down, while "maturation" is emphasized.[6] If schools adopt this notion, the results can be anticipated: teachers will be content to "wait" for readiness instead of taking steps to produce readiness.

9. These criticisms, admittedly inferential and theoretical in part, are less convincing than is research evidence as a basis for questioning the Committee's conclusions with respect to readiness. In the next section such research evidence is to be assembled, and the whole section may therefore be regarded as the ninth argument against the validity of the Committee's findings.

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[6] Experience and its effects are certainly recognized by the Committee in its use of foundations tests; but the Committee says little about the influence of prior experience as a factor in readiness. The criticism offered above is therefore directed against the Committee's rather consistent overemphasis on purely physiological "ripening."

EVIDENCE FROM OTHER LEARNING STUDIES

In the long run the best way to find out whether children are ready to learn anything in arithmetic or elsewhere, is to teach that thing and to observe the results. This is precisely what the Committee of Seven did. It is precisely what other investigators also have done, but in a different way. As has already been explained, none of the research studies to be considered here was specifically planned to secure evidence on readiness, but such evidence was forthcoming as a by-product. And, in the opinion of the writers, this evidence is no less valid than is that of the Committee of Seven.

Fowlkes (8), using specially prepared printed practice materials which permitted each child to detect and correct his errors at once, taught the one hundred combinations to thirty-one Grade IIIB children in twenty days. On the end test the average score was 91, and three fourths of the subjects wrote correct answers for seventy-nine or more of the facts. It is to be noted that the experimental period lasted four weeks, two weeks less than that of the Committee of Seven, and that one hundred instead of eighty-one combinations were taught. Yet Fowlkes's subjects showed a much higher level of achievement than did the Committee of Seven. It is true that no retention test was given six weeks later, but it is improbable that many of the children would have made retention scores lower than 80 per cent (the Committee's original standard), even without the advantage that might have accrued, had they had two weeks' longer study. Fowlkes's subjects seem to have been ready for all the combinations, hard as well as easy, though they were at the time in the first term of Grade III.

Washburne and Vogel (24), in their report on the relative difficulty of the combinations, mention in passing their success in teaching the combinations. Their subjects were 244 third-grade children, who studied materials devised by Washburne himself and designed to foster individual progress. No details of the experiment are available, but the average score on the one hundred combinations at the conclusion of teaching was almost 98 per cent. Why Washburne chose to disregard evidence which he had collected in 1924 in favor of the Committee of Seven data on readiness, it is difficult to understand.

Norem and Knight (14) taught the one hundred combinations to twenty-five superior third-grade children in a period of "about two or three months." While no data relating to final achievement are reported, apparently a satisfactory degree of mastery was attained. (Admittedly the children in this investigation were an unusually bright group. On this account, the significance of this experiment for the present purpose must be discounted.)

Brueckner (4) has reported data on the achievement of his 125 subjects, part of them from Grade IIB and the rest from Grade IIIB, in learning the facts for 2, for 2's, for 3, for 3's, for 4, and for 4's. The experiment ran for three weeks, one week being devoted to each two sets of facts (e.g., for 2 and for 2's). The final test showed approximately 98 per cent mastery of these particular facts, and comparable achievement for the other sets of facts. Be it noted that in the course of their instruction Brueckner's subjects learned fourteen facts which the Committee of Seven would now reserve to Grade VI (their products are larger than 20).

Wheat's subjects (26) were forty third-grade children who studied the thirty-nine supposedly most difficult combinations by means of devices which were intended to put meaning into the facts. These combinations, all of which would be deferred to Grade VI by the Committee of Seven, were learned satisfactorily with an average of twenty-one trials per combination.

The only investigator whose subjects showed relatively low mastery of the combinations taught were Wheeler's (27). After six weeks of drill these Grade III children could supply answers for only half the combinations, on the average, and one quarter knew only a third of them. Still, this level of mastery exceeded that of the subjects in the original Committee of Seven investigation.

Certain data from the investigations reported in Part II of this monograph are relevant at this point. In the local study (Chapter IV) half of the 124 Grade IIIA children made scores of 64 or higher, and three fourths made scores of at least 55. These scores represent, respectively, 79 per cent and 69 per cent of the possible score of 81. Yet, these children were tested on sixteen combinations which they had not been taught. Toward the end of Grade IVB, half the 117 subjects scored 80 or 81 (perfect) on the group test, and three fourths scored 75 or higher (93 per cent of the possible score). These experimental subjects, then, as measured by what they actually learned, must have been capable of learning the combinations at the time they were taught—and not merely the Committee of Seven's "easy" facts, but all the facts without 0.

The figures from the extended study (Chapter V) are comparable. Half the 557 subjects in Grade IIIA scored 68 or higher (84 per cent of the perfect score), and three fourths scored at least 61 (75 per cent). In Grade IVB, half of the 613 subjects scored 78, 79, 80, or 81, and three fourths scored at least 66 (81 per cent of perfect). These children too were ready for the combinations.[7]

One further point should be made with respect to the research findings which have just been summarized. It will be recalled that the Committee of Seven, in its statistical techniques, rejected all subjects who made unsatisfactory scores on the foundations test. The effect of this practice was to produce a selected sample, one whose learning should have been (though it was not) superior to average. No such selection took place in any of the other foregoing investigations, with the possible exception of that by Norem and Knight. Instead, the experimental children were typical of their grades. Yet, with these unselected subjects almost all investigators have reported superior achievement to that of the Committee's subjects. Here is further evidence of something extraordinary in the Committee's experimental situation, something which led to abnormally low achievement and hence to abnormally high MA standards for the grade placement of the combinations.

CONCLUDING STATEMENT

A serious word of caution is necessary in interpreting the research summarized in this chapter. For all the learning studies reviewed, the same measure of "learning" or "mastery" has been employed, namely, success in stating or writing correct answers for the combinations. This is the measure that the Committee of Seven adopted, and it seemed both fair and mandatory to use the same measure for the other learning studies, since the data from the latter were to be compared with the Committee data.

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[7] Certain data collected in the local study are relevant to the use of MA as a basis for predicting readiness. In Grade IIIA, which is closest to the first presentation of multiplication facts, accuracy scores on the group test correlate .45, .51, and .59, respectively with IQ, MA, and AA (Arithmetic Age, as determined from the new Stanford Achievement Test). General arithmetical competence correlates with learning slightly higher than does MA, and hence would seem to be superior for purposes of prediction. This advantage of course rests upon coefficients which are low in predictive value and upon a difference which is barely reliable; and the advantage might not have been claimed at all, were it not for Curr's findings in Scottish schools. Working with the process of long division, Curr found that MA was predictive of learning chiefly because it obscured the effects of IQ and that general arithmetical competence is a far better measure of readiness than is MA. William Curr, "Placement of Topics in Arithmetic," *Studies in Arithmetic* (Scottish Council for Research in Education; London: University of London Press, 1941), II, 183-218.

Persons who have read through the monograph to this point understand the inadequacy of correct answers as the sole criterion of learning and hence will want to enter objection to its use in determining readiness for the combinations. In none of the learning studies (except those reported in this monograph for the first time) have there been presented facts relating to the *processes* children use in securing their answers. When learning is described in terms of processes used, the problem of determining readiness is greatly complicated. One must then think of readiness, not for the combinations (all, or any part of them), but for this or for that way of thinking of particular combinations. To be concrete, one must recognize that children at some stage in their learning may be ready to find the answer for three 6's by counting, but not by solution, or by reversing the factors; or by solution, but not by using meaningfully habituated associations.

This conception of readiness is the one which is required by the data on processes presented in Part II, and it is the one which the writers propose. A change to this conception would result in the banishment of oversimplified notions of "readiness for the combinations" or of "readiness for these combinations." Acceptance of this conception would mean that we must cease thinking about readiness in all-or-none terms and in hard-and-fast categories; but it would also mean that teachers would be brought closer to learners and that they could then guide learning more intelligently and healthfully.

Part IV

Teaching the Multiplication Combinations

CHAPTER X

THEORETICAL AND PRACTICAL SUGGESTIONS RELATING TO INSTRUCTION

The purpose of this monograph is practical. Previously published research has been summarized, and new research (three separate studies) was organized, and the results thereof have been presented, all in the hope of improving methods of teaching the combinations. If at times the practical purpose has been lost to view, it has never actually been forgotten by the writers. Not infrequently criticism of research involves abstract and theoretical considerations in which, seemingly, the classroom, the teacher, and the learner have no place. But in the end the theorizing comes back to earth, and we have a sounder basis upon which to make changes in instructional practice.

As a matter of fact, this chapter, the most "practical" in the monograph, begins with theoretical considerations. To a greater extent than is always recognized,[1] the activities of the classroom spring from more or less abstract conceptions, even when these conceptions are not clearly identified and thoroughly verbalized. What the teacher does in arranging learning experiences is largely determined by his ideas (1) respecting the purposes of arithmetic in the elementary curriculum and (2) respecting the nature of the learning process, the way in which children realize the purposes of arithmetic. Accordingly, the first section of this chapter, "Theoretical Considerations," will be devoted to a discussion of these conceptions. The second major section will offer specific suggestions for the teaching of the combinations, but the "theoretical" section will have laid the foundation for these suggestions. Without this foundation the reasons for the suggestions would not be known, and their worth could not be easily appraised.

THEORETICAL CONSIDERATIONS

The Purpose of Arithmetic in the Elementary Curriculum

It is not a mere verbal trick to say that there is no such thing as arithmetic. Instead, there are arithmetic*s*, or several kinds of arithmetic. (1) As viewed by the mathematician, arithmetic is the science of number, and its content is selected without reference to social values and purely in terms of systematic mathematics. In teaching, emphasis is laid upon mathematical principles and relationships, and applications to "practical" everyday uses of number are disregarded.

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[1] Guy M. Wilson, "What Theory of Arithmetic?" *Educational Administration and Supervision,* XXV (Feb., 1939), 93-97.

(2) As viewed by the exponents of social utility, arithmetic is something else again. The content is composed of those concepts and skills which are essential to practical living. "Useless" topics, that is to say, skills and concepts that cannot be shown to function as such in the affairs of everyday life, are rejected, and in the process a great many understandings and principles dear to the mathematician are discarded. In the classroom the attempt is made to imbed each new concept or skill in a "natural situation" and to return the concept or skill, once taught, to use in connection with other "natural situations." As a matter of fact, sometimes only the arithmetic which occurs in "natural situations" is taught at all; all other arithmetic is eliminated as "traditional" and "nonfunctional." The degree of proficiency required with respect to the retained skills varies from school to school. Obviously, the activities of teachers and pupils alike will be quite different under conception (2) and under conception (1).

(3) A third kind of arithmetic may be called "textbook arithmetic." The content is whatever appears in the textbook; and teaching consists in seeing that children master that content by whatever amounts of drill are necessary. The net product is certainly not mathematical insight; nor is it insight into the quantitative aspects of our culture. Rather, it is likely to be a loose, incoherent assemblage of numerical tricks which are useful—and then only temporarily—for finding answers to textbook problems and examples.

There are other conceptions of arithmetic, but the list will be concluded with that to which the writers subscribe: (4) Arithmetic is a system of quantitative thinking which we teach to children so that they, now as children and later as adults, may live more efficiently, more richly, more intelligently, and more happily in our culture. This conception recognizes both a mathematical and a social aim for arithmetic.[2] The fundamental reason for teaching arithmetic is social, to enable children and adults to live more effectively in their environment; but the social aim cannot be achieved unless the mathematical aim is achieved as well. Attainment of the mathematical aim is one guarantee (but only one) that the social aim will also be attained.

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[2] A good statement of this conception is found in the following reference: R. L. Morton, "Introduction," *Arithmetic in General Education* (Sixteenth Yearbook of the National Council of Teachers of Mathematics; New York: Bureau of Publications, Teachers College, Columbia University, 1941), pp. 1-7.

This is not the place for an exhaustive comparison of the merits of the four conceptions of arithmetic. It must suffice to say: that conception (1) offers no assurance that children (and adults) will be sensitive to the quantitative aspects of their environment and hence will be able to *use* the mathematical concepts and skills which they learn; that conception (2), by omitting mathematical understandings and relationships which make arithmetic a system of quantitative thinking, can scarcely make children (and adults) quantitatively intelligent; at most it can make them efficient in the situations in which they have experienced number; and that conception (3) defaults on all counts: the products of learning are hardly worth the effort necessary to achieve them.

The conception which is here favored agrees with conception (1) that the mathematics of arithmetic is highly important, but disagrees with this conception in recognizing that the mathematics taught must be functional or useful in situations outside the arithmetic period. On the other hand, conception (4) differs from conception (2) in giving place to mathematical concepts and skills which, while they are not demonstrably "practical," are nevertheless essential to a meaningful grasp upon arithmetic as a system of thinking. The aim of conception (4) is to make learners *intelligent,* and not merely efficient, in their environment—more intelligent than adults and children now are—to the end that they may live *fully* in that environment, more fully than they now do. Those who expound conception (4) are not, in other words, content with the present level of arithmetical intelligence; on the contrary, they would raise this level by many degrees.

If intelligence in quantitative matters is accepted as an aim for the teaching of arithmetic, methods of instruction must be consonant with this aim. Concepts and skills which are "learned" unintelligently are scarcely available for intelligent use. It follows that at all stages children must *understand* what they learn; and this understanding is possible only to the degree that mathematical insight is developed. This requirement of instruction is not essential; indeed, it is not even important, to advocates of concept (3) and, in large degree, to advocates of conception (2).

The multiplication combinations *as facts* are sufficiently crucial to arithmetic, however arithmetic is conceived, to be included by all students of arithmetic in the subject matter of this area. But, even so, the various conceptions of school arithmetic affect considerably what will be done about the combinations. According to conception (1), the mathematical relationships inherent in the combinations are a part of the content, as are also the nature of the operation of multiplication and an understanding of the principles according to which factors are related and tables are organized. According to conception (2), these principles, relationships, and understandings are unimportant: they do not seem to "function" in life, and they may safely be disregarded.[3] These understandings, etc., meet a similar fate in the hands of exponents of conception (3), but not of conception (4). The last-named want the rationale of the multiplication combinations fully developed, but they also want learners to appreciate the value of what they learn and to have many experiences in using it.

Conceptions of the Learning Process

The conception a teacher holds with respect to the purposes of arithmetic largely determines how he teaches arithmetic. So does his conception of the learning process. Two quite different conceptions of learning are prevalent, and these are stated and compared in the following paragraphs.[4]

Learning through repetition.—What is probably the most popular view of learning, if one may judge from classroom practices, relies upon repetition for learning. The child is told what to do (but not why he does it), and then is given enough experience in doing that precise thing to "fix" the idea or skill. In the case of the multiplication combinations, he is told, for example, that three 6's are 18, and then he is practiced in writing and in saying the formula until the response "18" is immediately forthcoming upon the presentation of the stimuli ("three 6's").

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[3] They are disregarded by some writers on arithmetic who insist that they want arithmetic to be "meaningful." What these persons really mean is that they want arithmetic to be "significant," that is, to have social value. See B. R. Buckingham, "Significance, Meaning, Insight—These Three," *Mathematics Teacher,* XXXI (Jan., 1938), 24-30.

[4] For a more detailed analysis of these theories, see William A. Brownell, "Psychological Considerations in the Learning and the Teaching of Arithmetic," *Tenth Yearbook of the National Council of Teachers of Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1935), pp. 1-31. See also by the same author, "Two Kinds of Learning in Arithmetic," *Journal of Educational Research,* XXI (May, 1938), 656-664.

Concrete demonstrations are held to a minimum. For the learner's "discovery" of the relationship between three 6's on the one hand and 18 on the other hand is substituted the teacher's or the textbook's authority. From the outset the child is expected to think of "three 6's" precisely as does the adult, save that his thinking must be somewhat slower and less certain. The tables may not be taught, for if they are taught, the child is apt to solve unfamiliar combinations from combinations that are better known. On the contrary, it is felt that the combinations should be taught in random order, so that children will master each combination independently of the others and so learn precisely and only what they will later use in the way in which they will use it.[5] Roundabout procedures of all kinds are discouraged, for they are "detours" from the straight highway of learning: they keep the child from learning what he is supposed to learn, and the roundabout procedures may never be superseded by the automatic recall of combination answers. Teaching becomes the administration of drill, and learning consists in doing the same thing over and over and over again.

It may be argued that the foregoing statement is an exaggeration, that no one who relies upon repetition for learning is so extreme. The charge is admitted; the statement is an exaggeration, but only a slight exaggeration, as witness the following quotation from Wheeler (27):

> A child learns the multiplication facts solely to facilitate computation, and the ease of computation depends entirely upon the degree of mastery in terms of correct automatic response. The aim is to teach for one hundred per cent automatization of the number facts. Present methods of presenting the combinations increase rather than decrease the learning difficulties; they are logical but not psychological in approach. By introducing each multiplication fact by means of a concrete situation explained by additive counting ($3 + 3 + 3 = 9$), the association of adding is emphasized and suggests a "crutch" which children readily employ. . . . Associating the process of multiplication with addition suggests a counting-up the answer. It is too much to expect that a child, introduced to a combination through addition, will not use this method in computing an answer. In multiplication as well as in the other arithmetic processes some children develop an amazing skill in the use of "crutches." While this procedure makes learning the combinations very easy and may speed up initial learning, it hinders development of the rapid automatic response

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[5] This view of the learning process is especially well adapted to conception (3) above with regard to the purpose of arithmetic in the elementary curriculum. It is also evidently the view of learning which is accepted by many exponents of conception (2).

which is so essential for satisfactory progress with the larger combinations, with subsequent arithmetic processes, and for practical use in life. It is questionable whether introducing multiplication by additive concepts is any improvement over teaching by "tables"; both serve as aids in initial learning which have to be torn down and replaced by memorization before rapid and quick responses are possible. The basic psychological principles of learning are the same for all subjects, yet, while the alphabet method of teaching (202) reading is condemned, the same educators advocate a similar inductive method of teaching arithmetic. . . .

Learning the number combinations should be considered as a task of memorization, and the psychological principles for economy in memorizing should be applied even during initial instruction. Memorizing the combinations without the encumbrances of counting-up or adding may be a slower process in learning the easier combinations, but in the end it will mean a useful and immediate command of all the combinations. The time to present concepts of adding in multiplication is after the facts have been memorized, never before (203).

Learning through understanding.—Directly opposed to this view of the learning process is a second view which is here called "learning through understanding" in order to emphasize points of difference.[6] At one point the two views are in agreement: both views anticipate that the learner should be able promptly and invariably to supply the correct products for the combinations. Those who rely upon repetition would have this automatization established at once, and to them this is the sole end-product of learning. To those who stress understanding in learning, this end-product comes late, and it is but one of the outcomes which are sought in the case of the combinations. And the two groups of theorists differ completely as to the way in which this one end-product is achieved. What is "economical" learning to "repetition" theorists is "uneconomical" to "understanding" theorists, and vice versa.

It is held by "repetition" theorists that the use of addition, counting-up, and similar "crutches" makes for easy learning for the first combinations taught. How this can be true it is difficult to see. After all, nothing is much simpler than the memorization of a few short formulas like "two 4's are 8," "three 5's are 15," and the like. On the other hand, it takes time to develop the meanings which "understanding" theorists insist should be taught before the brief formulas. It is difficult also to see how repetition alone can make easy the learning of the last twenty or thirty combinations, for by this time the learner must be confused and incapable of taking on more such "learnings."

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[6] A number of students of arithmetic, among them competent psychologists as well as educators, hold to this second view with varying degrees of exactitude. Among these may be mentioned the familiar names: Bond, Buckingham, Buswell, McConnell, Morton, Sueltz, Thiele, and Wheat. For a fuller development of this view the reader is referred to their writings (particularly to T. R. McConnell, "Recent Trends in Learning Theory: Their Application to the Psychology of Arithmetic," *Arithmetic in General Education,* Sixteenth Yearbook of the National Council of Teachers of Mathematics; New York: Bureau of Publications, Teachers College, Columbia University, 1941, Chapter XI). Or, a recent monograph by the senior author may be consulted: William A. Brownell, *Learning as Reorganization* (Duke University Research Studies in Education, No. 3; Durham, N. C.: Duke University Press, 1939), pp. 69-87.

The source of this confusion is understandable. When combinations are memorized one after another, there is a progressive loss of "individuality" among them. There is, for example, increasing difficulty in isolating products over the extending scale of values. Thus, 63, the correct response for one combination, is so close to 64, the correct response for another, that the two facts cannot be kept separate in thinking. Other distinguishing characteristics disappear. For example, certain combinations may be temporarily easy to learn because their products for the first time involve the 30's, but with the learning of other combinations with products in the 30's, this special mark of distinction is gone. Again, some combinations, at first easily identified by the fact that their products are even numbers, later become confused with other combinations which have the same characteristic. Under conditions of purely repetitive learning, then, the learner tends to practice wrong responses, to give products which are correct for certain combinations but not for those being practiced. And this tendency is enhanced, as more and more combinations are memorized. On the other hand, when provided with aids for understanding, the learner fits the last combinations into an orderly series which is held together by interrelationships.

The fact of the matter is that children who are *taught* by "repetition" or drill, do not *learn* by "repetition." How are "repetition" theorists to explain the data presented in Part II of this monograph? Practically all the experimental subjects were taught the combinations by methods which correspond closely to those of drill. Yet, as was amply demonstrated, these children exhibited, to use Wheeler's term, an "amazing" proclivity to think of the combinations in terms other than those of the simple, brief formulas which they had been taught. Where did they get them?[7] From their teachers? But almost without exception the experimental teachers expressed surprise when shown the results of the interviews. Did the experimental children get them from other children? If so, where did *they* get them?

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[7] And where did Wheeler's subjects get them? For almost certainly his subjects used "crutches" and roundabout procedures. Under very comparable circumstances, among children whose instruction was purely that of drill, immature procedures were about as prevalent as among children who were taught

The explanation of immature thought processes under conditions of drill is not at all difficult to "understanding" theorists. Indeed, the use of such procedures would be expected even though energy were expended to forestall them. Neither children nor adults for long take kindly to brute memorization. When presented with disordered arrays of independent items, they adopt some means of "putting sense" into them. That is what children do when under conditions of drill they are required to master the multiplication combinations; and the immature procedures which they hit upon now and then are the means they have to relate and hold together the accumulating body of items to be learned. It is futile by prescribing more and more drill (repetition) to prevent the appearance of such procedures, just as it is futile to try to remove them later by more and more drill.[8] Children employ these procedures because they need them.

"Repetition" theorists are of course correct in holding that "the basic psychological principles of learning are the same for all subjects"—for the multiplication combinations as for all other topics in arithmetic. But this statement is likely to be misleading unless one keeps clearly in mind the *function* of principles. The "basic psychological principles" are intended to describe or explain what happens when one learns. It is one thing to say, "Whatever one learns, he learns according to the same psychological principles." It is quite another thing to say, "Whatever one learns, he learns in precisely the same way," that is, by engaging in the same behavior or by using the same process. The first statement is correct; otherwise, we do not have the *basic* psychological principles of explanation. The second statement is wholly incorrect, for as a matter of ordinary observation,

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meaningfully. [T. R. McConnell, "Discovery vs. Authoritative Identification in the Learning of Children," *Studies in the Psychology of Learning, II*, Studies in Education, Vol. IX, No. 5 (Educational Psychology Series, No. 2); Iowa City, Iowa: The State University of Iowa, 1934, pp. 13-62.] While Wheeler does not seem to have interviewed his subjects systematically, his statement that "children develop an amazing skill in the use of 'crutches'" is highly significant; and it warrants the inference that in spite of every effort to prevent immature procedures on the part of his subjects he was unsuccessful —to his "amazement."

[8] The effect of drill in the latter connection has been the subject of research. Drill merely causes children to become more proficient in the use of whatever process is employed at the start of drill. William A. Brownell and Charlotte Chazal, "The Effects of Premature Drill in Third-Grade Arithmetic," *Journal of Educational Research*, XXIX (September, 1935), 17-28.

we do not behave alike with respect to all kinds of learning tasks; nor do different persons behave alike in learning what is apparently the same thing.

In the case of meaningful learning, Gates has expressed the idea neatly and completely: ". . . great difficulty has come from the notion, often accepted in practice if not in thought, that a fact is a fact, and it will function without regard to *the way in which it is acquired.*"[9] Concretely, a grammatical rule may be memorized without understanding, or it may be first developed inductively and then mastered. And it makes a great deal of difference which way the rule is learned. When memorized without understanding, the rule will have little effect upon the learner's usage; when learned on a meaningful basis, some effect may reasonably be expected; hence, Gates's vigorous emphasis upon the importance of the way in which learning is conducted. There are, then, differences in the *process* of learning.

Gates rightly stresses the significance of these differences at the same time that he argues: "To explain the whole process of developing information and insight or perceiving facts of any kind requires no other fundamental principles than those we have been illustrating all along in connection with typewriting, memorizing, running a maze [and] learning a vocabulary."[10] In other words, the *explanatory principles* are identical for all learning. Yet, Gates does not for a moment suggest that the *process* is identical in all learning. On the contrary, as the quotation above clearly indicates, learning behavior (process) is recognized as varying with the particular task which faces the learner, so much so that Gates devotes separate chapters to some of the different kinds of learning.

The crucial error of "repetition" theorists lies precisely at this point: they fail to discriminate between the *principles of explanation* in learning and the *process* of learning, and they suppose that because the principles are alike, the process must be the same regardless of the learning task. Their next step is to accept as the prototype of all learning the process of rote learning or memorizing (to which of course the explanatory principles apply, as they must to all kinds of learning). Now, the learning task to which one responds by rote learning has unique characteristics: to the greatest extent possible all cues, all relationships, all chances to use effectively the results of previous experience are excluded, and the learner is left no alternative except to memorize. It is small wonder, then, that under these conditions the process of learning is seen to be that of memorizing.

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[9] Arthur I. Gates, *Psychology for Students of Education* (Revised Edition; New York: The Macmillan Co., 1930), pp. 328-329. Obviously "repetition" theorists accept this notion both "in thought" and "in practice."

[10] *Ibid.*, p. 360. Gates's principles (Readiness, Simultaneous Stimulation, and so on) are of course those which are implicit in his psychological system, according to which learning consists in the addition, elimination, and organization of "connections."

There is little place for rote learning in the school. Now and then of course arbitrary associations must be set up between ideas and symbols (for example, between the names of letters and the forms in which they are written and printed, between number ideas and the figures used to stand for them); but the occurrence of such learning tasks which children meet should regularly "make sense" to them; relationships with prior experiences should be readily identifiable, and, in a word, their learning should be meaningful.

According to "understanding" theorists, the multiplication combinations call for meaningful learning. Someone has suggested that a parrot can be taught to say, "Five 2's are 10"; but no parrot has been found able to use this "knowledge" in an intelligent way. The combination five 2's are 10 is much more than a series of sounds or of written words. To know and to be able to use the combination, the learner must:

(1) understand the meanings of the numbers as groups (2, 5, 10);
(2) know how to deal with successive groups of equal size in order to secure totals (the operation of multiplication);
(3) recognize in the form of the statement of the particular combination that he is to combine five groups which are equal in size, each made up of 2;
(4) be able to discover in many verbal and concrete situations that this relationship (of five 2's to 10) is involved;
(5) be able to think the product promptly, confidently, and accurately (without recourse to roundabout procedures, "crutches," or anything of the kind).

The words in which the first four statements are couched—"know," "understand," "discover," "recognize," "meanings"—these very words reveal how much learning in the case of the combinations transcends rote learning and how far memorizing falls short of meeting the needs of the situation. Overconcern with the fifth aspect of learning in the series above has blinded "repetition" theorists to the existence of the first four aspects. Yet, it is the first four aspects that "understanding" theorists emphasize, regarding the fifth as an end which is attained late in the course of learning. In the early stages of learning, immature processes are accepted, even encouraged, provided[11] only that they make the learning task sensible to the learner. All possible kinds of relationships (as in tables) are furnished—or, better, are discovered by the child himself under guidance. Principles and generalizations are developed and utilized, all in the hope of aiding the child to organize the task of learning the combinations. Gradually, the child is led to adopt more and more mature procedures until finally he achieves the ability to respond quickly, accurately, and confidently to the combinations, an ability which "repetition" theorists undertake to establish almost at the outset of learning.

"Understanding" theorists regard the ability to give products glibly as not worth developing; they know the perils of accepting facility in language as evidence of sound learning; they are therefore ready to wait for such facility until it is supported by a firm basis of understanding; and they know that meanings take time for their development, time which is filled with an abundance of varied experiences.[12] For them, the teaching of the combinations consists in providing these varied experiences and in guiding children as rapidly (but only as rapidly) as they can, to adopt more and more effective and mature ways of thinking of the combinations. The course of learning the combinations then resembles the picture of learning which has already been sketched in this monograph from the interview data collected from experimental subjects (Chapter IV, pp. 66-83, and Chapter V, pp. 88-93).

SUGGESTIONS FOR TEACHING THE COMBINATIONS

The following suggestions for teaching the multiplication combinations rest upon two notions or assumptions: (1) that arithmetic is a system of quantitative thinking which is taught so that children, as children and later as adults, may live more efficiently, more fully, more intelligently, and more happily in their environment; (2) that learning is a process of organizing and reorganizing experience. In the case of the multiplication combinations learning starts with relatively immature and inefficient thought procedures (the only kind children are then capable of) and proceeds by stages to the final state of meaningful and intelligent mastery, but employs at all times experiences and activities which are sensible to children. In the foregoing section of this chapter these two notions have been elaborated and defended. To the extent that these notions are unacceptable to the reader, the following practical suggestions will be equally unwelcome. As has been repeatedly stated, classroom practices are based upon the conception one entertains both with regard to the purposes of the subject matter and with regard to the nature of the learning process.

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[11] This proviso is important. Not all immature processes which children hit upon make for meaningful learning. It would be a mistake to encourage processes of this last kind. It would be a mistake too to infer that all children must adopt and use all possible processes and then abandon them in precisely the same order and at the same time in favor of more mature and economical processes.

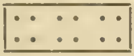
[12] A good, readable reference on the psychology of meaning is the chapter by T. R. McConnell entitled "The Development of Meanings" in Arthur I. Gates, *et al.*, *Educational Psychology* (New York: The Macmillan Co., 1942), chap. xiii.

Devices

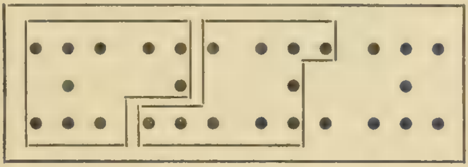
(1) The combinations should first be written in form (a), two 4's = 8, rather than in form (b), 2 × 4 = 8, or in form (c), $\begin{array}{r} 4 \\ \times\ 2 \\ \hline 8 \end{array}$. Form (a) carries its own meaning, whether read silently or stated orally: two groups of 4 each are to be combined, and this is just another way of saying the familiar "four and four are eight." When forms (b) and (c) are introduced, they should first be read, "two 4's are eight." The word "times" for "×" is not meaningful and should not be used until longer multiplication examples are taught in which the combinations occur. It is easier then to say "three times 426" and "forty-five times 93" than it is to say "three 426's" and "forty-five 93's"; but the transition to "times" should be made in this way in order to preserve and extend the original meaning.

(2) The tables should by all means be taught, but they should not be available at the outset and should never be "given" to children. Objections to the teaching of the tables rest upon the fact that some children recite the tables to secure answers for difficult combinations. As the data in Chapters IV and V showed, the extent of this practice has been exaggerated; but there is still no reason to encourage it. The argument for tables is that they help the learner to organize his task and they assist in the development of important meanings and generalizations.

When a new table is started, the combinations should first be presented in random order, the purpose being to enable children to discover the corresponding products (but not then to learn them). These preliminary experiences should illustrate again and again the nature of the operation of multiplication and should reveal the relationships between factors and product. Next, children should assemble the combinations in table order, observing how each product is increased over the preceding by the amount of the multiplicand (but not of the multiplier). Still later, on this meaningful basis children should learn the tables, but mixed practice should follow soon so that they will acquire mastery over the separate facts.

(3) Concrete devices should be used as needed, but the need should not be great in the middle of the third school year. Carper's data (pp. 123-127, above) show that at this time and prior to instruction on multiplication children readily deal with small numbers as abstract groups. For them, an occasional illustration or verification of a combination should suffice; such as  for three 4's = 12.

Wheat (26) has recommended the use of concrete groups for the harder combinations, but he would have the groups rearranged into units of 10 each. Thus, four 7's = 28 becomes



It is improbable that such experiences (they would come generally in Grade IV) are necessary as means of discovering and verifying all combinations; and it is also improbable that this device is needed to demonstrate further the decimal character of our number system. It is true that to secure 28 from four 7's we destroy the original groups and rearrange the objects into two standard units of 10 each, with 8 over. But this fact is implicit in the way in which the product is written: 28 means two 10's, with 8 over, and this fact needs only to be pointed out a few times for it to be understood and appreciated. A further objection to Wheat's device as a learning aid is its cumbersomeness. Let the reader draw the picture for nine 6's and alter it to show five 10's, with 4 over.

(4) There is plenty of room for drill in teaching the combinations: (a) drill on the abstract formulas as the competent adult uses these, and (b) drill on the generalizations and principles which make the combinations valid and relate them to each other. The danger in connection with (a) especially lies in prescribing drill too soon. Children are not ready for drill until they have a sound basis in understanding and can readily think of the combinations in abstract terms

Then drill increases facility of recall and fixes the combinations for retention. Introduced too soon, drill on the combinations (*if* children do what they are supposed to do) is subversive, for it leads to a substitution of verbal glibness for understanding, and with dire consequences. On this account, drill of type (b) above should be employed liberally before drill of type (a). Children need a good deal of experience in recognizing and applying generalizations and principles in order to develop the understanding upon which to base automatic mastery of the abstract combinations.

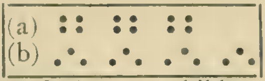
Meanings, Generalizations, Relationships, Principles

(1) The meaning of multiplication as a process should be taught, and taught prior to the teaching of the first combinations and again and again along with the teaching of other combinations. Multiplication is what we do with numbers when we want to find the total of several numbers which are equal in size. It is easier to find the total of thirty-six 97's by multiplication than by addition (which would require first the boresome writing of 97 thirty-six times before the labor of adding). Multiplication is a special case of addition, a short cut which we use in such instances as $9 + 9 + 9 + 9 + 9$, but not in such instances as $3 + 6 + 9 + 7 + 8$.

It follows that the multiplication combinations can profitably be developed through addition. Children readily see, as has already been stated, that two 4's are 8, because this is but a variation of the familiar $4 + 4 = 8$. They can as readily see that four 7's are 28 by adding in the example, $7 + 7 + 7 + 7$. The repeated discovery of products for new combinations by this device has the effect both of illustrating the meaning of multiplication and of proving the validity of products. "Repetition" theorists oppose this practice partly because they are not concerned about teaching the meaning either of multiplication or of the combinations, but more because they fear that children, once they have been exposed to the device, will always use it. This objection has already received attention.

(2) The principle governing reversals should be taught. Children should understand that the combinations three 4's and four 3's have the same product, and why this is so. The principle of reversals is that the order of the factors in a combination has no effect upon the product. But this principle does not mean that three 4's and four 3's are "the same." They are not, for in the first the basic group is 4 and this is taken three times whereas in the second the basic group

is 3, which is taken four times. A concrete representation of the two combinations brings out the difference. The principle needs to be *taught* because children may not discover it for themselves. After all, the pairs of combinations do not even sound alike, except to the sophisticated ear.



(3) Two principles help to teach the nineteen combinations in which 1 is the multiplier or the multiplicand: (a) When a number is multiplied by 1, the product is that number; (b) when 1 is multiplied by a number, the product is that number. The facts covered by these generalizations are almost self-evident to children, so self-evident that there is no need to teach these combinations as such until the digit 1 appears as part of a larger multiplier (e.g., 314) or multiplicand (e.g., 17). But though the facts are self-evident, there is value in bringing them together under covering statements.

(4) Similarly, the nineteen combinations with 0 as multiplier or multiplicand may be taught through establishing two principles: (a) When a number is multiplied by 0, the product is 0; (b) when 0 is multiplied by a number, the product is 0. These generalizations are readily intelligible if developed first in the form, two 0's = 0, three 0's = 0, etc.; and no 2's = 0, no 3's = 0, and so on. There is little point in teaching the facts with 0 as the multiplicand until two- and three-place multiplicands with 0 are used in examples, and in teaching the facts with 0 as multiplier until 0 occurs as part of a two- or three-place multiplier.

(5)[13] It helps if children see that all products end in even numbers when the multiplier is 2, 4, 6, or 8.

(6) It also helps if children realize that any product in which 5 is a factor must end in 5 or 0. (This relationship is usually a very easy one for children, a fact which leads some students of arithmetic to suggest that the tables for 5 and for 5's be taught immediately after the tables for 2 and for 2's.)

(7) The combinations with 9 as multiplicand have several interesting aspects: the digits in products always total 9; the first digit in products increases by 1, and the second figure decreases by 1 as the combinations are arranged in table form.

But there seem to be few comparable relationships among the majority of the combinations. Thiele (21) mentions the use of "generalizations" in teaching all combinations, but illustrates only with combinations in which 9 or 5 is a factor.

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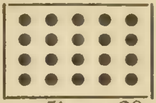
[13] Paragraphs 5 through 7 deal with number relationships which can scarcely be dignified by the term "generalizations," for they do not contribute much to meaning. These relationships call attention to properties of the combinations which enable children to discriminate among them, and so to identify them for easier learning.

(8) Except for the doubles, the multiplication combinations should be learned in pairs, e.g., five 8's = 40, eight 5's = 40. So learned, the total number of combinations is, in effect, reduced from 100 to 55. (The principle of reversals is of course the basis for understanding the pairing of facts.) Two cautions should be noted: (a) The practice of relating the paired facts should probably be postponed until after the first two or three tables. Premature introduction of the principle could overcomplicate the learning task, thus obscuring other essential relationships which are more readily grasped and which are sufficient in number and difficulty to engage the learner's full attention. (b) Care should be exercised to separate the paired facts later on for independent mastery.

(9) The pace of instruction must be adapted to the rate at which learning proceeds most effectively and most healthily. Two and more decades ago it was not uncommon to teach all the multiplication combinations in a period of a few months in Grade III, and in some of the experimental studies reviewed in the preceding chapter the learning period was cut to six and even to four weeks.

Within the last quarter of a century the practice has increasingly been to teach some limited number of combinations in Grade III and to postpone the rest to Grade IV. This change has resulted in part from the uncritical trend to move topics generally to higher grades (a trend which has already been discussed), in part from the intention to encourage more thorough learning of other arithmetical skills—or to extend them further—before starting multiplication, and in part from recognition of the fact that sound learning takes time. The last reason holds especially in the case of the multiplication combinations when the teaching task is made to include, besides the verbal statements themselves, the development of generalizations, principles, and understandings of relationships. When these are taught, the period of learning must be lengthened. Some courses of study wisely introduce the concept of multiplication and some of the easiest facts in the latter part of Grade II and do not expect mastery of all the combinations until two years later. During this long period children are given experiences with the combinations in wide varieties of ways, so as to increase meaning as well as to secure mastery. In other words, the program in such schools leaves nothing at all to time alone, for time is important to learning only to the degree to which it affords opportunity for experience.

(10) Once the process of division has been taught and a beginning has been made on the division combinations, the related multiplication and division facts may well be presented together. The drawing at the right shows four different facts. Which fact one sees depends upon the way in which one treats the groups, (a) by rows or by columns, and (b) by multiplication or by division. The four facts are: (a) four 5's = 20; (b) five 4's = 20; (c) 20 = four 5's, and (d) 20 = five 4's. The four facts together constitute "the whole story about 4, 5, and 20 in multiplication and division."[14] Frequent experience in "telling the whole story" in multiplication and division facts has the effect of emphasizing important relationships and of increasing ease of learning. Mixed drill, instituted later on, should then establish mastery of the separate combinations.



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[14] Actually "the whole story" here has two more parts, namely, (e) ¼ of 20 = 5, and (f) ⅕ of 20 = 4. The last two facts are obtained through partition, which is one of the two ideas in division (the other being quotition or true division). "The whole story" is held to four parts, however, in analogy with "the whole story" in addition and subtraction.

APPENDIX

TABLE 19

EXTENDED STUDY, GRADE IIIA: PROCESSES USED WITH SELECTED COMBINATIONS AS REVEALED IN INTERVIEWS (N = 155 — 474)*

Process	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7 6	9 2	4 7	5 3	8 6	7 9	3 6	*Total*
H....	201	89	165	221	94	201	223	59	117	303	167	281	100	71	230	2,522
M....	87	42	100	94	23	103	100	31	40	88	88	97	38	26	96	1,053
G....	33	60	32	23	46	30	20	18	44	4	37	7	53	37	21	465
S.....	44	24	42	35	15	54	31	12	27	30	55	24	36	14	36	479
C.....	28	5	35	24	5	26	23	3	7	18	35	20	14	4	28	275
R.....	24	37	67	23	13	24	60	12	17	20	58	25	15	34	50	469
T.....	17	14	21	14	12	24	12	11	19	7	19	13	12	15	8	218
V.....	5	1	2	2	2	4	2	2	3	2	4	3	5	2	2	41
N....	6	14	7	8	7	6	3	5	14	2	10	3	14	13	2	114
I......	..	1	1	1	1	1	..	1	..	..	1	..	1	1	1	10
X....	1	1	2	1	1	1	..	1	..	..	..	1	..	2	..	11
*Total**	446	288	474	446	219	474	474	155	288	474	474	474	288	219	474	5,667

*Totals vary among the combinations for the reason that only interview data are included for combinations which had been supposedly carried to mastery in learning. A total of 140 of the Grade IIIA pupils had finished the multiplication tables; 15 more had completed all except the one product for 9 × 9 (that is, had studied through the 8's table, with reverses); 64 more had studied through the 7's; 69 more through the 6's; 158 through the 5's; and 28 through the 4's only.

TABLE 20

EXTENDED STUDY, GRADE IVB: PROCESSES USED WITH SELECTED COMBINATIONS AS REVEALED IN INTERVIEWS (N = 483)

Process	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7 6	9 2	4 7	5 3	8 6	7 9	3 6	*Total*
H....	273	177	222	297	172	263	247	222	218	353	206	345	208	172	273	3,648
M....	84	63	53	72	61	62	66	84	62	67	56	73	64	61	58	986
G....	19	60	27	18	102	27	14	56	63	3	41	5	68	72	17	592
S.....	21	21	25	14	26	31	14	19	26	15	26	8	29	31	14	320
C.....	17	20	30	13	31	21	19	11	23	16	24	12	29	19	15	300
R....	21	59	102	33	22	27	101	27	27	16	91	20	16	46	84	692
T.....	32	13	18	28	17	37	17	14	17	11	19	18	21	22	12	296
V.....	2	1	1	2	..	4	..	1	2	1	..	1	..	..	1	16
N....	12	41	4	5	44	4	3	43	40	..	14	1	42	55	5	313
I......	..	..	..	..	6	4	2	4	2	1	3	..	3	2	1	28
X....	2	28	1	1	2	3	..	2	3	..	3	..	3	3	3	54

TABLE 21

EXTENDED STUDY, GRADE IVA: PROCESSES USED WITH SELECTED COMBINATIONS AS REVEALED IN INTERVIEWS (N = 425)

*Process**	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7 6	9 2	4 7	5 3	8 6	7* 9	3 6	*Total*
H....	313	231	244	292	247	291	259	262	260	330	240	335	262	235	275	4,076
M....	56	46	45	61	53	53	50	58	62	52	39	50	59	58	41	783
G....	5	18	18	7	30	14	8	18	20	5	11	4	22	25	8	213
S.....	7	11	10	11	19	14	6	11	12	5	17	5	16	12	5	161
C.....	5	8	5	4	7	6	3	3	7	3	12	1	8	8	3	83
R....	16	71	90	28	28	25	86	38	26	21	85	23	20	51	83	691
T.....	20	33	11	19	32	20	12	26	30	8	20	5	31	27	9	303
V.....	2	..	2	3	..	2	1	1	..	1	..	2	1	1	..	16
N....	1	6	..	..	9	..	..	8	8	..	1	..	5	7	..	45
I.....	..	1	..	..	..	..	..	..	..	..	..	..	1	..	1	3

*"X" is omitted, since there was but one instance of this process, with 9 × 7.

TABLE 22

EXTENDED STUDY, GRADE VB: PROCESSES USED WITH SELECTED COMBINATIONS AS REVEALED IN INTERVIEWS (N = 426)

*Process**	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7 6	9 2	4 7	5 3	8 6	7 9	3 6	*Total*
H....	275	210	208	284	232	271	233	244	252	310	208	315	266	203	246	3,757
M....	100	89	74	82	106	88	72	110	91	82	81	72	95	89	61	1,292
G....	13	26	18	10	34	7	8	22	23	3	27	3	15	37	12	258
S.....	9	15	9	6	12	17	11	11	17	2	16	8	13	17	9	172
C.....	5	8	8	8	8	4	4	5	5	6	10	6	3	7	7	94
R.....	16	53	89	24	13	26	91	20	13	12	62	10	17	49	75	570
T.....	4	14	16	10	16	11	5	7	15	10	19	9	9	15	13	173
N....	..	5	1	..	1	1	..	2	6	..	..	1	3	6	1	27
I.....	4	5	3	2	4	1	2	5	2	1	2	2	5	3	2	43
X....	..	1	..	..	..	..	..	..	2	..	1	..	..	..	..	4

*There were no cases of visualization (V); hence this category is omitted.

TABLE 23

EXTENDED STUDY, GRADE VA: PROCESSES USED WITH SELECTED COMBINATIONS AS REVEALED IN INTERVIEWS (N = 373)

Process	7 5	6 9	3 8	6 5	8 7	6 4	3 7	9 8	7 6	9 2	4 7	5 3	8 6	7 9	3 6	*Total*
H....	277	217	216	268	226	260	250	237	235	286	215	295	245	206	249	3,682
M....	68	67	61	63	65	60	44	77	71	66	48	45	68	65	52	920
G....	4	16	5	4	20	1	4	10	12	..	11	..	6	24	3	120
S.....	5	20	19	7	29	12	7	21	23	4	27	4	16	18	10	222
C.....	4	2	3	5	6	5	3	..	4	3	6	4	4	3	9	61
R.....	9	35	59	20	12	21	57	20	15	8	54	19	16	38	47	430
T.....	6	10	8	6	10	10	6	6	10	5	11	6	14	12	3	123
V.....	..	..	..	..	..	1	..	..	1	..	..	..	..	1	..	3
N....	..	2	..	..	2	1	..	1	2	1	1	..	1	4	..	15
I.....	..	1	1	..	2	1	1	..	..	..	..	..	2	..	..	8
X....	..	3	1	..	1	1	1	1	..	..	..	..	1	2	..	11

TABLE 27

COMPARISON OF BOYS AND GIRLS IN SAMPLES DRAWN FROM LOCAL STUDY

Character, with Measure	*Grade IIIA*		*Grade IVB*		*Grade VB*	
	Boys	*Girls*	*Boys*	*Girls*	*Boys*	*Girls*
CA.....Q_1	8-11	8-10	9- 6	9- 5	10- 4	10- 3
Md	9- 2	9- 3	9-10	9-10	10- 9	10- 9
Q_3	9- 9	9- 7	10- 3	10- 3	11- 2	11- 3
Q	5.0	4.5	4.5	5.0	4.0	6.0
MA.....Q_1	8- 8	9- 1	9- 7	9- 2	10- 6	10- 2
Md	9- 6	9- 7	10- 2	10- 2	11- 4	11- 2
Q	9-10	10- 3	10- 9	10-11	11-10	11-11
Q_3	7.0	7.0	7.0	11.5	8.0	10.5
AA.....Q_1	8- 3	9- 0	9- 9	9- 3	10-10	10- 5
Md	9- 5	9-11	10- 3	9-11	11- 2	11- 1
Q_3	10- 1	10- 3	10- 7	10- 4	11- 5	11- 5
Q	11.0	9.5	5.0	6.5	3.5	6.0
IQ......Q_1	88	93	95	92	98	94
Md	102	106	103	104	105	106
Q_3	110	114	115	114	111	114
Q	11.0	10.5	10.0	11.0	6.5	10.0

GENERAL INSTRUCTIONS

[*Directions to Teachers*]

The purpose of this letter is to give you certain general information concerning the investigation on the learning of the multiplication combinations. Very soon now we shall send you the actual testing materials which you are to use, together with directions for giving the tests; but right now we want you to know why we are making this study and how it is to be conducted.

PURPOSE OF THE STUDY

There have been surprisingly few research investigations relating to the learning of the multiplication combinations. By contrast, the number of experimental studies in the case of the addition and the subtraction combinations is very much larger; and consequently we now are better equipped to teach these combinations. As a matter of fact, even the few studies dealing with multiplication have been less helpful than they might have been, for the data collected have provided little information on *how children learn* the combinations; instead, the primary concern has been *how well* they learn the combinations. We cannot hope to improve very much our instruction on the multiplication combinations until we have information on children's processes and procedures in learning.

It is our firm belief that this study will be of real value to all teachers who participate. We are confident that they will secure important new insights into their pupils' difficulties in mastering the multiplication combinations—insights which should result in much more effective teaching. What we are trying to say is that the uses of the data will not be limited to the preparation of a formal printed report which will appear in some professional journal; rather, the data should be highly suggestive to every teacher who co-operates in securing them.

PLAN OF THE STUDY

It is planned to test children in Grades IV and V before December 1 and again next spring about May 1. In Grade III, the tests will be given only in the spring, since in very few schools are many of the multiplication combinations presented during the first term or semester.

The testing program consists of two parts: (1) group tests on the multiplication facts and (2) individual tests for selected children. The group tests (1) will be given to *all* children and will include all the multiplication combinations usually taught in the third grade. The individual tests (2) will be given by interviews and will contain only fifteen combinations. The method of choosing pupils for tests (2) will be described in a later communication. The schedule of testing, then, is as follows:

	Before December 1	*About May 1*
Grade III	....	Group tests—all pupils Individual tests—selected pupils
Grade IV	Group tests—all pupils Individual tests—selected pupils	Same as "Before December 1"
Grade V	Same as in Grade IV	Same as "Before December 1"

From the group tests we shall secure the usual measures of learning, namely, (1) number of correct responses and (2) rate of work. When the group test blanks are sent you, they will be accompanied with instructions for administration so as to yield the two measures desired. From the rate and accuracy scores for the several hundred children in this study it will be possible to construct curves showing the progress of learning in half-grades, from the second term of Grade III through the second term of Grade V.

The group test data will be of value, as you can see; but they will be of less value than the interview data. From the individual tests we hope to secure data which have never before been collected—data which reveal how the children of the successive half-grades *think* about the combination and how their thought processes change from time to time. It is from our records of your pupils' thought processes that we hope to discover *how* children learn the combinations.

The group tests, you will find, will be easy to administer, for you have had many experiences in giving similar tests. The individual tests, on the other hand, are not easy to give. On the surface, the interview appears to be a very simple procedure for securing research information; in reality, it is a very difficult procedure to employ effectively. This is especially true in investigations such as this one, in which one seeks to get at children's mental processes and manipulations.

The interviewer should observe not only the subject's *verbal responses,* but also his *facial expressions, lip movements,* and *mechanical activities,* since all of these may reveal what the child is thinking. A normal, friendly relationship should exist between interviewer and subject. The interviewer must be careful, however, to *avoid approval or disapproval* and the *giving of cues or suggestions.* (The child is apt to adopt whatever suggestion is offered in place of his true process.) Also, the interviewer must exercise caution *not to push the questioning* too far, lest the child become self-conscious, and his responses and methods cease to be typical.

These warnings may seem to be quite unnecessary, but actually they are not. Teachers who have had the most experience with the interview will be the first to recognize the need of the cautions and to advise their less experienced friends to heed them. Every possible safeguard must be taken to guarantee *true records* of children's thought processes in dealing with the interview combinations. You will soon discover that an apparently "known" combination may not be "known" at all, but that the child has recited the tables, or guessed, or solved from a better-known combination, or counted, or given a glib, meaningless answer, even if it be correct. It is precisely this kind of information which the interview is designed to supply. You see, for the purposes of this part of the investigation, the correctness or the incorrectness of the answer is relatively unimportant: what we want to know is how the child *thinks* to get his answer.

ADDITIONAL CAUTIONS

We assure you that we are in no way trying to determine the effectiveness of your instruction. That is to say, we are not attempting to compare groups or classes of pupils or to compare school systems. For this reason you need feel no embarrassment at the showing your pupils make; we honestly do not know what a good showing is; rather, we hope to learn this from our study. Particularly is this true in the matter of the mental processes discovered by means of the interview. Do not be surprised if some of your "better" pupils count combinations you thought they knew; "better" pupils actually do this. Merely give us a complete and an accurate account of what you find. When we have assembled all the facts, we shall be more able to say what we can properly expect of children at the various grade levels.

One last caution: Please do not revolutionize your methods of teaching the multiplication combinations merely on the basis of the results from the first tests. Too hasty conclusions from incomplete data may actually be detrimental. Our advice is to postpone drastic changes until after you have full information from all grades in your school and from both series of tests (December and May). You will still have some days or weeks in which to put this new information to its most profitable uses.

REPORTS OF RESULTS

Please arrange to give the group tests as near December 1 as is feasible, and follow them at once with the interviews, which should be completed within two weeks of the group tests. Similarly, please plan to give the spring group tests as near the first of May as is practicable, and to complete the individual tests before May 15.

It will take us some time to tabulate and to interpret the results from the separate co-operating centers. Because of this, it will be impossible to report to you what we have found before the winter of 1942. It is our expectation to print our findings rather fully either as a monograph or as an article in a professional periodical. In either case, we shall furnish you a copy.

TESTING MATERIALS

We shall send your co-ordinator enough group tests for all of your pupils and enough interview blanks for the reports of the individual testing. You may use the first group tests and the first interview blanks as you wish, but we want them in our hands before the

Christmas holidays. Please do not place any marks on the test blanks, lest we be unable to understand and interpret them.

A LAST WORD

May we take the last paragraph to tell you how very much we appreciate your co-operating in this study? It goes without saying that without the help of teachers far removed from the Duke University campus, we should be quite unable to make this investigation. We realize that we are asking you to donate a portion of the small amount of time that is left you from your usual teaching activities. We believe that your efforts will be repaid to you personally by what you learn and that they will also contribute more widely to a greatly improved teaching of the multiplication combinations throughout the country in general.

THE GROUP TEST

Purpose. The purpose of the group test is to secure the traditional kinds of information with regard to knowledge of the multiplication combinations. This knowledge is measured in terms (1) of accuracy (number of correct answers) and (2) of rate (time required to finish). The directions below are designed to secure both of these measures.

Combinations included. Eighty-one of the 100 multiplication combinations are included in the test, the 0-combinations being omitted since they seem to constitute a separate and special problem. The inclusion of some facts for 7's, 8's, and 9's will work a hardship on certain co-operating third grades, for the reason that these facts are not taught until the fourth grade. However, it is hoped that even in these classrooms the group test will be administered precisely as it is intended to be given. As has already been explained, there is to be in this study no comparing of class with class or school with school and no publishing of data which will permit of such comparisons. The directions for the test will make it perfectly clear to pupils that all are not expected to have perfect papers.

Scoring. Use the test papers as you see fit, provided that you make no marks on them. The papers will be scored at Duke University, and at this time the last two items on the test papers (for rate and accuracy) will be entered by us.

Schedule. The group tests are to be given in Grades IV and V as near December 1 as possible, and are to be repeated in these grades and given for the first time in Grade III about May 1. In each case the test papers are to be sent to the co-ordinator within two weeks of the testing (for the first tests, before the Christmas holidays *without fail*).

Cautions. Please follow exactly the directions for administering the tests; otherwise, the results will not be uniform from class to class. Errors may easily be made at two points. First, be sure to *stress equally* the idea of rate and of accuracy of work. The children should not hurry so fast that they miss combinations they usually get, nor should they dawdle needlessly long over their work. Second, be careful about the timing. The 30-second intervals (see below) must be exactly preserved.

Note to co-ordinators. In returning the test materials (group and individual) please use the Railway Express Agency, sending the package collect. This method will be cheaper than the use of the U. S. mails (the materials would have to come first class), and the cost of returning the package will be automatically taken care of at this end.

We are sending at this time enough group tests and Interview Blanks for *all* the testing, both now and in the spring. If by chance you find that you need more copies of either, please let us know at once. You understand of course that in the December testing the third-grade pupils are not to be included. The group test papers and the Interview Blanks for the first testing (Grades IV and V) are to be sent to Durham, North Carolina, before the holidays.

DIRECTIONS FOR ADMINISTERING THE TEST

1. Provide each pupil with *two* sharpened pencils before starting the test.
2. Distribute the papers *face downward*, and direct the pupils not to turn the blanks until they are told to do so.
3. *Read* the following directions:
   "This is a test on the multiplication facts (or combinations, whichever word you are accustomed to use). You are to write the answer for each fact (combination) on the line immediately below it. At the right you will find space for work if you need it.
   "When I tell you to begin, turn the paper over and begin to write the answers. *Work as rapidly as you can without making mistakes.* If you find facts (combinations) which you do not

know, go right on to the next ones. Do not stop too long on the hard facts (combinations).

"Several times while you are working, I shall say MARK. When you hear MARK, draw a line under the example you are working on and keep on with the test. Understand: MARK does not mean to stop; it means to mark the example you are working on.

"When you have finished the test, write your name at the bottom of the page where you see NAME. Then turn your paper over, and wait quietly for the rest to finish.

"Are there any questions?" (If so, answer them freely.)

4. Say "MARK" exactly 30 seconds after you give instructions to begin the test, and repeat "MARK" every 30 seconds thereafter. Be careful—*especially* careful—with the timing, since the rate scores are to be derived from the pupils' underscoring of examples.
5. Allow enough time for all except the very slowest three to five pupils to finish. These pupils will be those who obviously could not finish because they do not know the combinations. Of course, if *all* your pupils know the facts, allow time for all to finish.
6. Before taking up the papers, have the pupils supply the following information at the bottom of their test papers:

   (1) Name of school
   (2) Grade
   (3) City (or county)
   (4) Your name, as teacher

   Items (5) and (6) will be filled in at Duke University.

THE INDIVIDUAL TEST

Purpose. The purpose of the individual tests is to obtain information concerning children's thought processes while securing answers to multiplication combinations. For each child interviewed you are to record this information by code letter (described below) on the Interview Blank for your class.

Combinations included. Only fifteen combinations are included in the individual test, for the most part those which usually cause most trouble. These selected combinations are listed at the top of the Interview Blank and are to be presented one at a time in the order given.

Scoring. For each combination enter in the appropriate space: (1) the child's numerical answer (as 4, 16, etc.), whether this is correct or incorrect; (2) the code letter which corresponds to the child's processes as he reports them to you; (3) in certain cases the exact words of the child. The instances in which (3) is required are those in which you are unable to classify these processes according to the code symbols.

Schedule. The individual tests are to be completed within two weeks of the group tests, and the Interview Blanks should be handed to the co-ordinator with the group test blanks.

Cautions. Please read again the section of the GENERAL INSTRUCTIONS in which the difficulties of interviewing are described. We cannot emphasize too much the need of care in questioning children in order to secure a true record of their processes without influencing them by questioning.

Selection of pupils for individual tests. Please arrange to interview as many of your pupils as you can—the more, the better. For purposes of planning you may estimate each interview to require between twelve and fifteen minutes.

If you find it impossible to interview every pupil in your class, begin with the first pupil on your *alphabetical roll.* Then continue with the other pupils, following the alphabetical list.

If it happens that a child who is chosen for an interview is repeating the grade, indicate this fact by placing *R* in front of his name on the Interview Blank. Also, enter *A* beside *R* for each child whose failure was due wholly or in large measure to weakness in arithmetic.

Code for processes. Study the following classification of children's processes very carefully. In all, eleven code symbols are listed and described, eight of which (all except *N, I,* and *X*) represent rather commonly found procedures on the part of children. It should be understood, however, that *these eight particular procedures are not the only possibilities.* They simply are processes which were discovered in a preliminary investigation, and you may expect to find others that should be reported in full.

The eleven code symbols follow:

Code Letter	*Description of Class*
H	*Habituation, meaningful.* The correct answer is confidently given at once, *with every indication of understanding.*
M	*Memory, rote.* The answer is given at once and with apparent confidence, *but there is no evidence of understanding.*
G	*Guessing.* The answer, usually wrong, is given rather promptly, but it is evident that the child is guessing.

S *Solution.* The child starts with a familiar combination, and adds to, or subtracts from, the product to get the answer. There are many kinds of solution, for example:

$3 \times 4 = 2 \times 4 = 8, + 4 = 12$
$3 \times 4 = 4 \times 4 = 16, - 4 = 12$

C *Counting.* The child adds or counts *the same unit* several times.

$3 \times 4 = 3, 6, 9, 12$; or $4, 8, 12$.

R *Reversal.* The child interchanges multiplier and multiplicand, in order to get a more familiar order.

$6 \times 3 = ?$, $3 \times 6 = 18$, so $6 \times 3 = 18$

T *Tables.* The child starts with a lower combination in the particular table involved and recites the table to the required point.

$4 \times 6 = ?$ $4 \times 1 = 4$; $4 \times 2 = 8$; $4 \times 3 = 12, \ldots$
or, $4 \times 4 = 16$; $4 \times 5 = 20$; $4 \times 6 = 24$.

V *Visualization.* The child reproduces groups of objects or figures in clear imagery and works with these.

$4 \times 5 = ?$ 4×5c (nickel) $= 20$c, or 20.

N *No attempt.* The child makes no attempt to state the answer.

I *Indeterminate.* The child gives an answer for the combination, but you cannot ascertain *how* he got it (that is, his processes).

X *Miscellaneous.* The child reports a process not included in the preceding list, or you are uncertain as to its proper classification. Report what the child said, using the back of the Interview Blank if this is necessary.

The largest amount of difficulty will arise in distinguishing among *H*, *M*, and *G*, all of which cover responses usually given rather quickly. The purpose of using the three classes is to try to differentiate between combinations which are known meaningfully and those which have merely been memorized without understanding. The interviewer must be sole judge on this point, but it is wise to guard against too ready classification of responses under *H*. As a rule, the best basis for decision is the general bearing and behavior of the child: his confidence, casual remarks, and the like.

In case of doubt it is sometimes helpful to tell the child that his correct answer is wrong—that $4 \times 6 = 25$, not 24, as he has said. If the child can demonstrate in any convincing way that 4×6 is 24, not 25, then he should be given credit for *H*.

On the other hand, too liberal use of this device will naturally lead the child to employ with later combinations processes which he otherwise would not have used. For this reason the interviewer should use this device only sparingly, relying for the most part on her knowledge of the child and on her observations of his behavior during the interview.

DIRECTIONS FOR INDIVIDUAL TESTS

1. Before starting the interview write the combinations on separate cards, in the order on the Interview Blank.
2. Select a time and place that is free from both physical and mental distractions. If possible, you and the subject should be alone.
3. Seat the child at a small table opposite you, the better to observe all kinds of significant behavior.
4. Expose the first combination card and record the child's numerical answer (right or wrong) in the appropriate space.
5. Then question the child carefully to determine what method or methods he used in arriving at his answer. The following questions are suggestive:

   a. What were you thinking when you said — × — equal — ?

   b. What did you do to find the answer for — × — ?

   c. How do you know that — × — equal — ?

   If the child does not volunteer answers to these questions, encourage him to further effort by some absurd illustration, such as (for 6 × 5): "Did you think 30 because 5 and 6 are 30?" Record on the Interview Blank the symbols which correspond to the method or methods.
6. In the case of processes which you must record as *X* (miscellaneous) please remember to report the processes in full, so that they may be classified at Duke University.
7. Continue with the other combinations in the same way.

MULTIPLICATION TEST

										Space for work
1.	1 6	4 4	2 9	5 1	4 8	9 6	5 3	3 7	5 4	
2.	6 5	8 8	1 5	6 4	3 3	8 2	3 4	9 5	1 8	
3.	5 6	7 2	3 8	7 5	1 4	6 8	9 3	2 6	8 9	
4.	8 4	3 7	6 1	5 9	8 3	2 2	5 7	9 1	3 5	
5.	1 7	[illegible] 5	6 2	4 9	8 6	1 3	3 6	5 5	3 2	
6.	4 6	7 9	1 1	9 2	4 7	4 3	5 8	6 9	2 5	
7.	7 8	5 2	7 1	2 8	6 6	2 3	8 1	9 8	7 4	
8.	4 2	2 1	9 4	6 7	3 9	4 5	8 7	6 3	2 4	
9.	1 9	9 7	3 1	7 7	7 3	1 2	9 9	7 6	4 5	

Name____

(1)____________________ (2)____________________

(3)____________________ (4)____________________

(5)______________ (6)______________

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